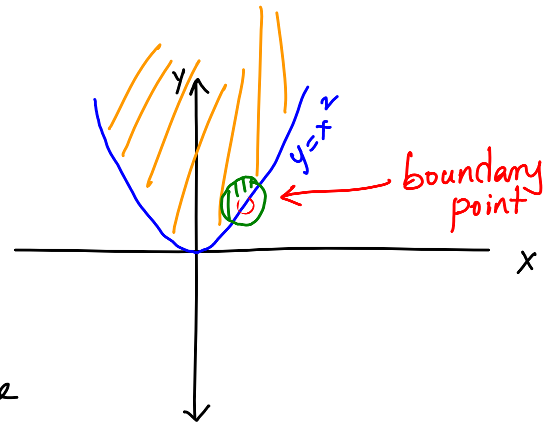
MATH 273 – Lecture 3 (09/02/2014)

Open v/s closed sets – another example.

$R: \; y \geq x^2$

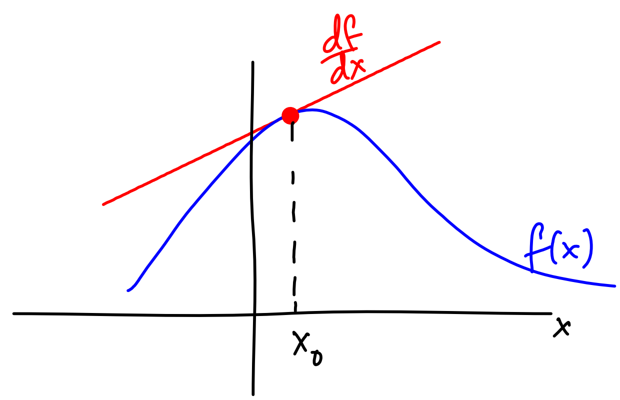


R is not open, as all points in R are not interior. Points on $y=x^2$ are not interior points.

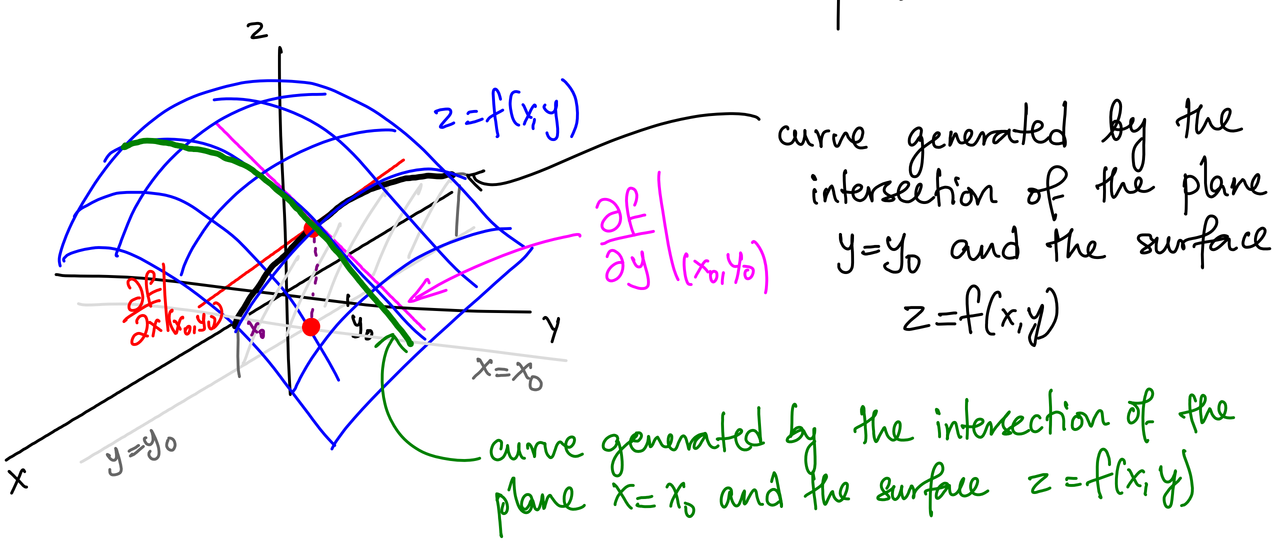
In fact, the points on $y=x^2$ are the boundary points. Since R contains its boundary points, R is closed.

Also, R is unbounded here.

In 1D, derivative of $f(x)$ at $x=x_0$ is the tangent to the curve $y=f(x)$ at $x=x_0$.



The intuition carries over to higher dimensions, one variable at a time.



Prob 27 (Section 13.3)

Find f_x, f_y, f_z where $f(x,y,z) = \sin^{-1}(xyz)$.

Recall that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f_x \text{ or } \frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(xyz)^2}}\frac{\partial}{\partial x}(xyz)$$

$$= \frac{yz}{\sqrt{1-x^2y^2z^2}}$$

$$f_y = \frac{1}{\sqrt{1-(xyz)^2}}\frac{\partial}{\partial y}(xyz) = \frac{xz}{\sqrt{1-x^2y^2z^2}}$$

In the exams, you will be given such complicated derivatives.

Similarly, $f_z = \dfrac{xy}{\sqrt{1-x^2y^2z^2}}$.

Prob 35 (Section 13.3)

$f(t,\alpha) = \cos(2\pi t - \alpha)$. Find the partial derivatives of f with respect to each variable.

$$f_t = \frac{\partial f}{\partial t} = -\sin(2\pi t-\alpha)\cdot\underbrace{\frac{\partial}{\partial t}(2\pi t-\alpha)}_{2\pi - 0} = -2\pi\sin(2\pi t-\alpha)$$

α is constant here

t is constant here

$$f_\alpha = \frac{\partial f}{\partial \alpha} = -\sin(2\pi t-\alpha)\cdot\underbrace{\frac{\partial}{\partial \alpha}(2\pi t-\alpha)}_{0-1} = -\sin(2\pi t-\alpha)\cdot(-1)$$

$$= \sin(2\pi t-\alpha).$$

This problem illustrates that we could use any symbol to represent variables, e.g., α and t here, and not just x, y, z, etc.

Implicit Differentiation

Rather than specifying $y=f(x)$ (i.e., the form of the function) explicitly, we could have an equation satisfied by x and y describing the function.

e.g., $\quad xy + x^2 = 0$

To find $\frac{dy}{dx}$, we differentiate the equation with respect to x (both sides of '=').

$$\frac{d}{dx}\left(\underbrace{xy}_{\text{product}}+x^2\right) = \frac{d}{dx}(0) = 0$$

as $y=f(x)$ here

$$1\cdot y + x\frac{dy}{dx} + 2x = 0,$$ and solve for $\left(\frac{dy}{dx}\right)$

$$x\left(\frac{dy}{dx}\right) = -(2x+y). \quad \text{So} \quad \frac{dy}{dx} = \frac{-(2x+y)}{x}.$$

We can do similar implicit partial differentiation in higher dimensions.

Prob 65 (Section 13.3) Find $\frac{\partial z}{\partial x}$ at the point $\underset{x,\,y,\,z}{(1,1,1)}$ given

$$xy + z^3x - 2yz = 0, \quad z=f(x,y),$$ and that the partial derivative $\frac{\partial z}{\partial x}$ exists at $(1,1,1)$.

We differentiate the equation partially with respect to x.
Thus, y is kept constant, but $z=f(x,y)$, and hence is a function of x.

$$\frac{\partial}{\partial x}\left(xy + \underbrace{z^3 x}_{\text{product}} - 2yz\right) = \frac{\partial}{\partial x}(0) = 0$$ y is constant here

$$y\frac{\partial}{\partial x}(x) + x\frac{\partial}{\partial x}(z^3) + z^3\frac{\partial}{\partial x}(x) - 2y\frac{\partial}{\partial x}(z) = 0$$

$$y + x\left(3z^2\frac{\partial z}{\partial x}\right) + z^3 - 2y\left(\frac{\partial z}{\partial x}\right) = 0$$

plug in $(x,y,z) = (1,1,1)$

$$1 + 1.3(1)^2\left(\frac{\partial z}{\partial x}\right) + (1)^3 - 2(1).\left(\frac{\partial z}{\partial x}\right) = 0$$

$$1 + 3\left(\frac{\partial z}{\partial x}\right) + 1 - 2\left(\frac{\partial z}{\partial x}\right) = 0 \qquad \text{giving } \frac{\partial z}{\partial x} = -2.$$

$$2 + (3-2)\left(\frac{\partial z}{\partial x}\right) = 0$$

Second order partial derivatives

If we differentiate $f(x,y)$ twice, we get

$$\frac{\partial^2 f}{\partial x^2}, \ \frac{\partial^2 f}{\partial y^2}, \ \frac{\partial^2 f}{\partial x \partial y}, \text{ and } \frac{\partial^2 f}{\partial y \partial x}, \text{ or } f_{xx}, f_{yy}, f_{yx}, f_{xy}.$$

Order of x,y : "start from inside" → this is an easy to remember rule of thumb. The "inside" is where 'f' sits".

So, $$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{or} \quad f_{yx} = (f_y)_x$$

inside inside

Similarly, $$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \text{or} \quad f_{xy} = (f_x)_y.$$

Prob 42 (Section 13.3)

$f(x,y) = \sin xy$. find all second order partial derivatives of f.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\cos(xy)\cdot\frac{\partial(xy)}{\partial x}\right) = \frac{\partial}{\partial x}\Big(\underbrace{y\cos(xy)}_{\frac{\partial f}{\partial x}}\Big) = y\cdot\frac{\partial}{\partial x}(\cos(xy))$$

$$= y\cdot -\sin(xy)\frac{\partial}{\partial x}(xy) = -y^2\sin(xy).$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\cos(xy)\frac{\partial}{\partial y}(xy)\right) = \frac{\partial}{\partial y}\Big(\underbrace{x\cos(xy)}_{\frac{\partial f}{\partial y}}\Big) = x.-\sin(xy)\frac{\partial}{\partial y}(xy)$$

$$= -x^2\sin(xy).$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\Big(\underbrace{x\cos(xy)}_{\text{product}}\Big) = x\cdot\frac{\partial}{\partial x}\cos(xy) + \overset{\frac{\partial}{\partial x}x}{1}.\cos(xy)$$

$$= x - \sin(xy).y + \cos(xy)$$

$$= \cos(xy) - xy\sin(xy)$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\Big(\overbrace{y\cos(xy)}^{\frac{\partial f}{\partial x}}\Big) = 1\cdot\cos(xy) + y\cdot\frac{\partial}{\partial y}\cos(xy)$$

$$= \cos(xy) - xy\sin(xy)$$

Notice that $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x}$. Does this result always hold?

Under certain conditions, yes! More on this result in the next lecture.