MATH 273 – Lecture 4 (09/04/2014)

We noticed that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ in the last example from Lecture 3.

When does this result hold?

The mixed derivative theorem

If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are all defined throughout an open region containing a point (a,b), and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Applies to higher order derivatives as well, e.g., $f_{xxy} = f_{xyx} = f_{yxx}$

Prob 55 (Section 13.3) Which order will calculate f_{xy} faster – first x or first y? Answer without actually finding the derivatives.

(d) $f(x,y) = y + x^2 y + 4y^3 - \ln(y^2+1)$

x first, since x appears only in one term of f.

To check, $\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x.$

Prob 61 $f(x,y) = 2x + 3y - 4$. Find the slope of the line that is tangent to the surface $z = f(x,y)$ at $(\underset{x}{2}, \underset{y}{-1})$

(a) lying in the plane $x = 2$. $\frac{\partial f}{\partial y}$ at $(2,-1)$

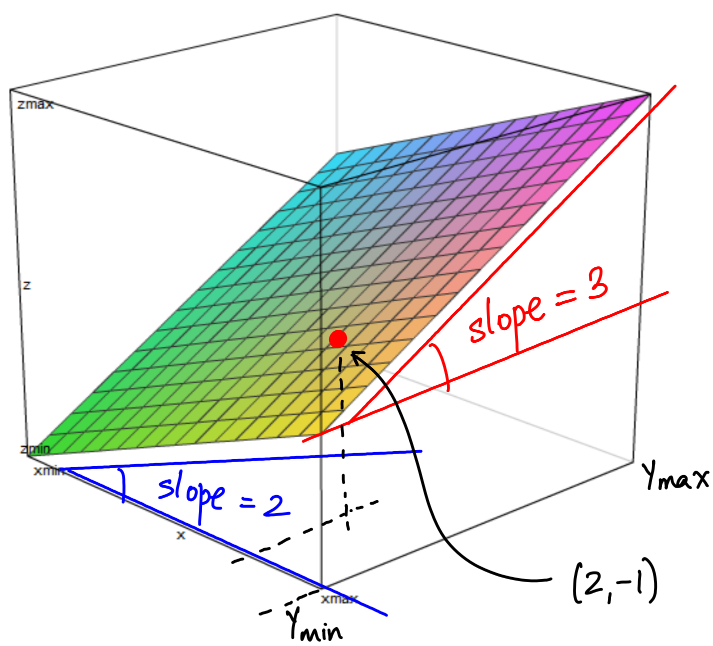
(b) lying in the plane $y = -1$. $\rightsquigarrow \frac{\partial f}{\partial x}$ at $(2,-1)$

(a) $\frac{\partial f}{\partial y} = 3$. (b) $\frac{\partial f}{\partial x} = 2$

Recall from Lecture 3 that $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ is the tangent to the surface at (x_0,y_0) lying in the plane $y=y_0$. Notice that $y=y_0$ is a plane parallel to the xz-plane.

In this case, the surface is actually a plane – notice that all terms are linear – $2x$ and $3y$. Indeed, the partial derivatives in this case at $(2,-1)$ are the slopes of this plane at that point. Since it's a plane, these two slopes are the same irrespective of which point you are considering!

Here is the surface from online 3D grapher:



The Laplace Equation (Still section 13.3)

The 3D Laplace equation is satisfied by the steady state temperature distribution in space. If $T = f(x,y,z)$, then

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \quad \text{or} \quad f_{xx} + f_{yy} + f_{zz} = 0$$

In 2D, we get $f_{xx} + f_{yy} = 0$.

Prob 73 (Section 13.3) Show f satisfies the Laplace equation.

$f(x,y,z) = x^2 + y^2 - 2z^2$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x) = 2 \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(2y) = 2 \qquad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}(-4z) = -4$$

Hence $f_{xx} + f_{yy} + f_{zz} = 2 + 2 + (-4) = 0.$

Prob 78 $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$. Show f satisfies the Laplace equation.

Recall: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

$$f_{xx} = \frac{\partial}{\partial x}\left[\frac{1}{\left(1+\frac{x^2}{y^2}\right)}\frac{y^2}{y^2}\cdot\frac{1}{y}\right] = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{y\cdot(-1)\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

y is constant

$$y\frac{\partial}{\partial x}\left((x^2+y^2)^{-1}\right) = y\cdot(-1)(x^2+y^2)^{-2}\cdot 2x$$

$$f_{yy} = \frac{\partial}{\partial y}\left[\frac{1}{\left(1+\frac{x^2}{y^2}\right)}\cdot x\,\frac{-1}{y^2}\right] = \frac{\partial}{\partial y}\left(\frac{-x}{x^2+y^2}\right) = -x\frac{\partial}{\partial y}\left[(x^2+y^2)^{-1}\right]$$

$$\left(1+\frac{x^2}{y^2}\right)y^2$$

$$= -x(-1)(x^2+y^2)^{(-2)}\cdot 2y$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

So, $f_{xx} + f_{yy} = 0.$

The Chain Rule (Section 13.4)

In 1D, $w = f(x)$ and $x = g(t)$, and if f is differentiable w.r.t (with respect to) x, and g is differentiable w.r.t t, then w is differentiable w.r.t t, and

$$\frac{dw}{dt} = \frac{dw}{dx} \cdot \frac{dx}{dt}$$ (Chain rule for one independent variable t and one intermediate variable x)

We generalize to higher dimensions, i.e., we consider more than one independent variable as well as more than one intermediate variable.

Theorem 5. Chain rule for one independent variable and two intermediate variables.

$w = f(x,y)$ is a differentiable function of x and y, and $x = x(t)$ and $y = y(t)$ are differentiable functions of t, then $w = f(x(t), y(t))$ is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \quad \text{or}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Notice how the partial derivatives are appearing – for the function f of multiple intermediate variables. The derivatives w.r.t to the single independent variable t are not partial.

Prob 2 (13.4) $w = x^2+y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$

Express $\frac{dw}{dt}$ using (a) chain rule and (b) by first writing w as a function of t, and then finding $\frac{dw}{dt}$.

(a) Chain rule: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$

$x = \cos t + \sin t$

$$= (2x)\underbrace{[-\sin t + \cos t]}_{y} + (2y)\underbrace{[-\sin t - \cos t]}_{-(\cos t + \sin t) = -x}$$

$$= 2x[y] + 2y[-x] = 2xy - 2xy = 0.$$

(b) $w = x^2+y^2 = (\cos t + \sin t)^2 + (\cos t - \sin t)^2$

$$= 2\underbrace{(\cos^2 t + \sin^2 t)}_{1} = 2\cdot 1 = 2$$

$$\frac{dw}{dt} = 0.$$