MATH 273 – Lecture 5 (09/09/2014)

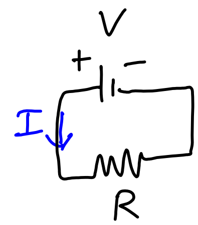
Next week (September 16 & 18): Class will meet in **VECS 125** ↳ via AMS Videoconferencing

Office hours on Skype (ID: **wsucomptopo**).

Chain rule $w=f(x,y)$, $x=x(t)$, $y=y(t)$, all functions differentiable.

Then $$\frac{dw}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

Problem 41 (Section 13.4)



V: voltage, I: current, R: resistance

$V=IR$

Voltage decreases with time, as the battery weakens.
Resistance increases with time as the resistor heats up.
How does the current change with time?

Find $\frac{dI}{dt}$ when $R=600$ ohms (Ω), $I=0.04$ amps, $\frac{dR}{dt}=0.5$ ohms/sec, $\frac{dV}{dt}=-0.01$ volts/sec.

$V=IR$. Apply chain rule: $$\frac{dV}{dt}=\frac{\partial V}{\partial I}\cdot\frac{dI}{dt}+\frac{\partial V}{\partial R}\cdot\frac{dR}{dt}$$

I, R are functions of time, and hence V is also one.

$$-0.01 = R\cdot\left(\frac{dI}{dt}\right)+I\left(\frac{dR}{dt}\right)$$

$\frac{\partial V}{\partial I}=\frac{\partial(IR)}{\partial I}=R\cdot 1=R.$

i.e., $$-0.01=600\left(\frac{dI}{dt}\right)+\underbrace{0.04(0.5)}_{0.02}$$

This gives $-0.01-0.02=600\left(\frac{dI}{dt}\right)$. Hence $\frac{dI}{dt}=\frac{-0.03}{600}=-5\times10^{-5}$ amps/sec.

We can extend the chain rule to 3 intermediate variables.

Theorem 6 $w = f(x,y,z)$ is differentiable, and x, y, z are differentiable functions of t. Then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\,\frac{dz}{dt}.$$

Prob 6 $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$.

Find $\frac{dw}{dt}$ using (a) chain rule, and (b) directly, and find $\frac{dw}{dt}$ at $t=1$ in each case.

(a) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (0 - y\cos xy)\cdot(1) + (0 - x\cos xy)\left(\frac{1}{t}\right) + (1-0)\,e^{t-1}$$

at $t=1$, $x = t = 1$, $y = \ln t = 0$, $z = e^{1-1} = 1$.

$$\left.\frac{dw}{dt}\right|_{t=1} = (-0\,\cancelto{0}{\cos 0})(1) + (-1\cdot\underbrace{\cos 0}_{1})\left(\frac{1}{1}\right) + 1\cdot e^{1-1}$$

$$= 0 - 1 + 1 = 0.$$

(b) $$w = z - \sin xy = e^{t-1} - \sin(t\ln t)$$

product rule $\frac{d(t)}{dt}\cdot\ln t + t\cdot\frac{d}{dt}(\ln t)$

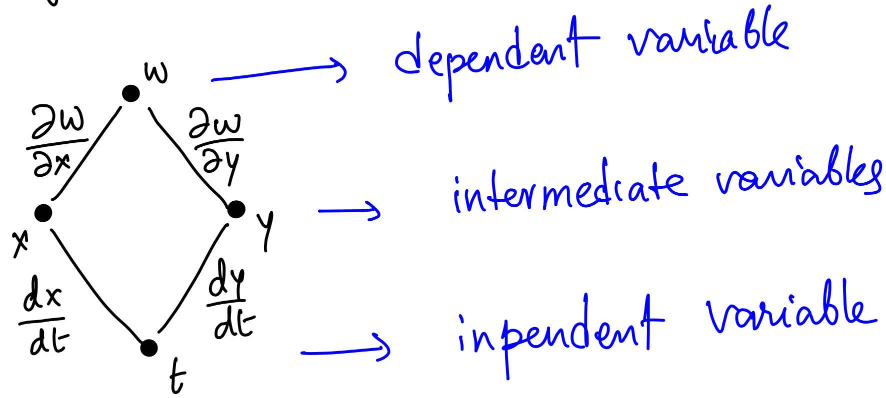
$$\frac{dw}{dt} = e^{t-1} - \cos(t\ln t)\cdot\left[1\cdot\ln t + \cancel{t}\cdot\frac{1}{\cancel{t}}\right]$$

$$\left.\frac{dw}{dt}\right|_{t=1} = e^{1-1} - \cos(1\cdot\underbrace{\ln 1}_{0})\,[1\cdot 0 + 1] = 1 - 1\times 1 = 0.$$

Branch Diagrams

We present a pictorial way to easily decipher the expressions for derivatives using the chain rule. The idea is not to memorize the expressions by heart!

$$w = f(x,y), \quad x = x(t), \quad y = y(t).$$



Multiply the terms along each branch, and add to get the expression for $\frac{dw}{dt}$.

$$\frac{dw}{dt} = \underbrace{\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}}_{\text{left}} + \underbrace{\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}}_{\text{right}}$$

There are three levels – one each of the dependent, intermediate, and the independent variables. We draw a point, or a "node", for each variable in the corresponding level. Then we draw a branch, or a line, from each node to the nodes in the next level.

If there is only one branch going down from a variable node, the corresponding derivative is the usual one, i.e., not a partial derivative. For instance, we have $\frac{dx}{dt}$ and $\frac{dy}{dt}$ above. When there are two or more branches going down, each branch gets a partial derivative term.

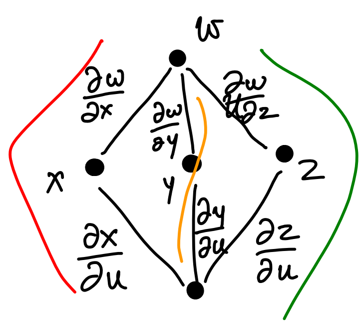
Prob 15 Draw branch diagram for $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ when

$w = h(x,y,z)$, $x = f(u,v)$, $y = g(u,v)$, $z = k(u,v)$.

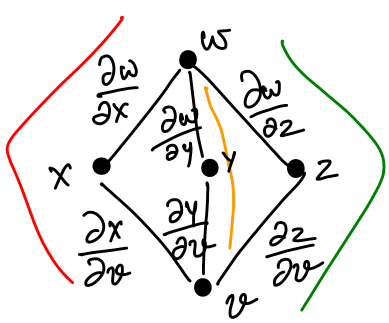
Note: This problem presents essentially the same result that is specified in Theorem 7. It is much easier to draw the branch diagrams to figure out the expressions for the partial derivatives!

Notice that w is the dependent variable, x,y,z are the intermediate variables, and u,v are the independent variables here.

Since there are two independent variables, w is ultimately a function of both of them (u,v here), and hence we get partial derivatives w.r.t to each of them. We draw separate branch diagrams for each of them as well.



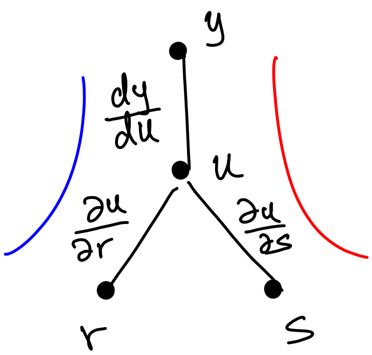
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u}$$



$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}$$

We do **not** add the two partial derivatives! We write them separately. Later on, in Section 13.6, we will learn how to consider linear combinations of such partial derivatives in order to define a tangent plane at a given point.

Prob 20 Branch diagram $\frac{\partial y}{\partial r}$ for $y = f(u)$, $u = g(r, s)$



$$\longrightarrow \frac{\partial y}{\partial r} = \frac{dy}{du} \cdot \frac{\partial u}{\partial r}$$

$$\longrightarrow \frac{\partial y}{\partial s} = \frac{dy}{du} \cdot \frac{\partial u}{\partial s}$$

Here, we have 1 dependent variable y, 1 intermediate variable u, and 2 independent variables r and s. As such, y is ultimately a function of r and s, thus giving rise to two partial derivatives.