MATH 273 – Lecture 6 (09/11/2014)

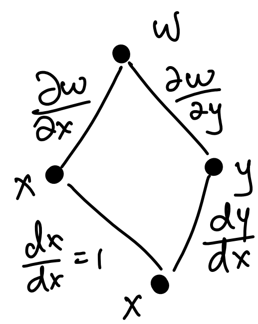
Chain rule (continued..)

<u>Implicit differentiation</u> $F(x,y)=0$ defines y as differentiable function of f. We differentiate both sides of the equation, and solve for $\frac{dy}{dx}$. We extend this idea to multiple variables and use the chain rule in the process.

<u>Prob 25</u> $x^3-2y^2+xy=0$ defines y as a differentiable function of x. Find $\frac{dy}{dx}$ at $(1,1)$.

Let $w=F(x,y)=x^3-2y^2+xy$. Then $w=0$ is the given equation.

Differentiate w.r.t x (both sides):



$$\frac{dw}{dx}=\frac{\partial w}{\partial x}\underbrace{\frac{dx}{dx}}_{=1}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dx}=0$$

So, $$(3x^2+y)\cdot 1+(-4y+x)\frac{dy}{dx}=0$$

Plug in $(x,y)=(1,1)$ to get

$$\underbrace{(3(1)^2+(1))}_{4}+\underbrace{(-4(1)+(1))}_{-3}\frac{dy}{dx}=0.\quad \text{So}\quad \frac{dy}{dx}=\frac{4}{3}$$

<u>Prob 31</u>. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at (π,π,π) when
$x\ y\ z$

$F(x,y,z)=\sin(x+y)+\sin(y+z)+\sin(z+x)=0$ defines z implicitly as a differentiable function of x and y.

Note: x and y are independent variables, z is dependent on both x and y.

Differentiate partially w.r.t. to x $\quad F(x,y,z)=0$

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1}+\frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial x}}_{=0}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x}=0$$

So, $$[\cos(x+y)\cdot 1+0+\cos(z+x)\cdot 1]\times 1+0+[\cos(y+z)+\cos(z+x)]\left(\frac{\partial z}{\partial x}\right)=0$$

$$[\cos(x+y)+\cos(z+x)]+[\cos(y+z)+\cos(z+x)]\left(\frac{\partial z}{\partial x}\right)=0.$$

Plug in $(x,y,z)=(\pi,\pi,\pi)$ to get

If $[\cos(y+z)+\cos(z+x)]=0$ then $\frac{\partial z}{\partial x}$ is not defined

$$[\cos(2\pi)+\cos(2\pi)]+\underbrace{[\cos(2\pi)+\cos(2\pi)]}_{\neq 0}\left(\frac{\partial z}{\partial x}\right)=0$$

Hence, $\quad 2+2\left(\frac{\partial z}{\partial x}\right)=0$ giving $\frac{\partial z}{\partial x}=-1.$

Now, w.r.t. y, we get

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial y}}_{=0}+\frac{\partial F}{\partial y}\cdot\underbrace{\frac{\partial y}{\partial y}}_{=1}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}=0$$

$$0+[\cos(x+y)\cdot 1+\cos(y+z)\cdot 1+0]\times 1+[\cos(y+z)+\cos(z+x)]\left(\frac{\partial z}{\partial y}\right)=0$$

Plug in $(x,y,z)=(\pi,\pi,\pi)$ to get

$$[\cos(2\pi)+\cos(2\pi)]+[\cos(2\pi)+\cos(2\pi)]\left(\frac{\partial z}{\partial y}\right)=0$$

$$2+2\left(\frac{\partial z}{\partial y}\right)=0, \quad \text{i.e. } \frac{\partial z}{\partial y}=-1.$$

The book gives details of the implicit function theorem, which specifies when we can solve for $\left(\frac{\partial z}{\partial x}\right)$ in this fashion — as long as $\frac{\partial F}{\partial z}\neq 0$.

<u>Prob 43</u> $f(u,v,w)$ is differentiable, and $u = x-y$, $v = y-z$, $w = z-x$.

Show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$.

f : dependent

u, v, w : intermediate

x, y, z : independent

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}$$
$$= f_u\cdot(1-0) + f_v\cdot(0-0) + f_w\,(0-1)$$
$$= f_u - f_w.$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial y}$$
$$= f_u\cdot(0-1) + f_v\cdot(1-0) + f_w\,(0-0)$$
$$= -f_u + f_v$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial z}$$
$$= f_u\,(0-0) + f_v\,(0-1) + f_w\,(1-0)$$
$$= -f_v + f_w$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = (\cancel{f_u} - \cancel{f_w}) + (-\cancel{f_u} + \cancel{f_v}) + (-\cancel{f_v} + \cancel{f_w})$$
$$= 0.$$

Notice that we never used the exact form of $f(u,v,w)$ in this problem!

Direction derivative and Gradient Vector (Section 13.5)

We saw that $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ gives the slope of the line tangent to the surface $z=f(x,y)$ at (x_0,y_0) in the plane $y=y_0$. $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}$ was interpreted similarly. We now extend the idea of taking derivatives to arbitrary directions – and not just along x or y.

For a simple analogy, consider the function which gives the height of a mountain, $h=f(x,y)$. At any point on the mountain, the height may change differently in different directions.

Climbers on Mt. Rainier:

large negative slope – drop into crevasse!

tangent is steep (slope is large)

near zero slope – flat terrain (locally)

From where the climber is standing, the slope of the function (height) is very large looking to the right. It is negative looking to his left.

We define the directional derivative of a function f along a direction specified by the unit vector $\hat{u}$ at point $P_0 = (x_0, y_0)$ essentially in a similar fashion to how we have been defining derivatives so far using limits.

In 1D $\quad \left.\frac{df}{dx}\right|_{x_0} = \lim_{s \to 0} \frac{f(x_0+s) - f(x_0)}{s}$

$f(x,y)$; $\quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{s \to 0} \frac{f(x_0+s, y_0) - f(x_0, y_0)}{s}$

To generalize to considering derivatives in arbitrary directions, we think about $f(\cdot)$ as a function taking a vector as input.

Let $\bar{p} = \begin{bmatrix} x \\ y \end{bmatrix}$ (the vector with 2 components x and y, or, $\bar{p} = x\hat{i} + y\hat{j}$);

and we consider the directional derivative of f at $\bar{P}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ in the direction $\hat{u}$ (unit vector).

lower case letters with "bar" indicate vectors

$$\left. D_{\hat{u}} f \right|_{P_0} = \lim_{s \to 0} \frac{f(\bar{P}_0 + s\hat{u}) - f(P_0)}{s}$$