MATH 273 – Lecture 7 (09/16/2014)

Recall: directional derivative of $f(x,y)$ at $P_0(x_0,y_0)$ in the direction $\hat{u} = u_1\hat{i} + u_2\hat{j}$ ($\hat{u}$ is a unit vector, i.e., $\underbrace{\|\hat{u}\|}_{\text{length}} = 1$) is

$$\left(D_{\hat{u}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(P_0 + s\hat{u}) - f(P_0)}{s}$$

With $\bar{P}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = x_0\hat{i} + y_0\hat{j}$, we have $\bar{P}_0 + s\hat{u} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + s\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$= \left(x_0\hat{i} + y_0\hat{j}\right) + s\left(u_1\hat{i} + u_2\hat{j}\right)$$

$$= (x_0 + su_1)\hat{i} + (y_0 + su_2)\hat{j}$$

$$\left(D_{\hat{u}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(\overbrace{x_0+su_1}^{x}, \overbrace{y_0+su_2}^{y}) - f(x_0,y_0)}{s}$$

$x = x_0 + su_1 = x(s)$

$y = y_0 + su_2 = y(s)$

When $\hat{u} = \hat{i}$ (i.e., the x-direction), $\hat{u} = \underbrace{1}_{u_1}\cdot\hat{i} + \underbrace{0}_{u_2}\cdot\hat{j}$, then

$$\left(D_{\hat{i}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(x_0+s, y_0) - f(x_0,y_0)}{s} = \frac{\partial f}{\partial x}$$

Similarly, when $\hat{u} = \hat{j}$ (i.e., the y-direction), $\hat{u} = 0\cdot\hat{i} + 1\cdot\hat{j}$, then

$$\left(D_{\hat{j}}f\right)_{P_0} = \lim_{s\to 0} \frac{f(x_0, y_0+s) - f(x_0,y_0)}{s} = \frac{\partial f}{\partial y}.$$

How to compute $\left(D_{\hat{u}} f\right)_{P_0}$?

We have $f(x,y)$ with x,y being functions of s. We apply chain rule to compute $f(x,y)$

$$x = x_0 + s u_1$$
$$y = y_0 + s u_2$$
$$\frac{dx}{ds} = u_1, \quad \frac{dy}{ds} = u_2$$
$$\hat{u} = u_1 \hat{i} + u_2 \hat{j} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\left.\frac{df}{ds}\right|_{\hat{u}, P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \underbrace{\frac{dx}{ds}}_{u_1} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \cdot \underbrace{\frac{dy}{ds}}_{u_2}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0} \hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \hat{j}\right]}_{\text{gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1 \hat{i} + u_2 \hat{j}\right]}_{\hat{u}} = \begin{bmatrix} \left(\frac{\partial f}{\partial x}\right)_{P_0} \\ \left(\frac{\partial f}{\partial y}\right)_{P_0} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Def The **gradient vector** (or, just gradient) of $f(x,y)$ at point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \left.\left(\frac{\partial f}{\partial x}\right)\right|_{P_0} \hat{i} + \left.\left(\frac{\partial f}{\partial y}\right)\right|_{P_0} \hat{j},$$

where the partial derivatives are evaluated at $P_0(x_0, y_0)$.

"gradient of f"
"del f", "grad f"

Hence the directional derivative of f along $\hat{u}$ (unit vector) at $P_0(x_0, y_0)$ is

Scalar product

$$\left(D_{\hat{u}} f\right)_{P_0} = \left(\nabla f\right)_{P_0} \cdot \hat{u}$$

<u>Prob 2</u> $f(x,y) = \ln(x^2+y^2)$, $P_0(1,1)$. Find gradient of f at P_0, Sketch the gradient and level curve passing through P_0.

$$\left(\nabla f\right)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} \hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0} \hat{j} = \left(\frac{2x+0}{x^2+y^2}\right)_{P_0} \hat{i} + \left(\frac{0+2y}{x^2+y^2}\right)_{P_0} \hat{j}$$

$$= \frac{2}{2}\hat{i} + \frac{2}{2}\hat{j} = \hat{i} + \hat{j}.$$

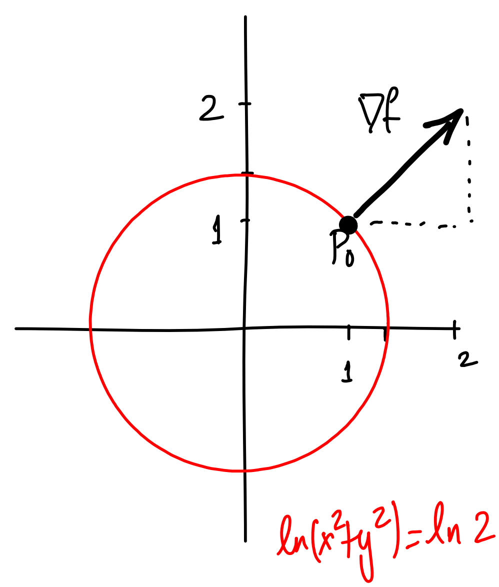
$f(x,y) = \ln(x^2+y^2)$

at $P_0(1,1)$, $f(1,1) = \ln(1^2+1^2) = \ln 2$.

The level curve is $f(x,y) = \ln 2$

$$\ln(x^2+y^2) = \ln 2$$

i.e., $x^2+y^2 = 2$



Definition of ∇f naturally extends to 3 (or higher) dimensions: for $f(x,y,z)$,

$$\nabla f\Big|_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\hat{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\hat{j} + \left(\frac{\partial f}{\partial z}\right)_{P_0}\hat{k}$$

<u>Prob 7</u> $f(x,y,z) = x^2+y^2-2z^2+z\ln x$, $P_0(1,1,1)$, find $(\nabla f)_{P_0}$.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(2x + \frac{z}{x}\right)\hat{i} + (2y)\hat{j} + (-4z+\ln x)\hat{k}$$ plug in (1,1,1)

$$= \left(2\times 1 + \frac{1}{1}\right)\hat{i} + (2\times 1)\hat{j} + (-4(1)+\ln 1)\hat{k} = 3\hat{i}+2\hat{j}-4\hat{k}.$$

Equivalently, $\nabla f = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$.

Prob 13 $g(x,y) = \frac{x-y}{xy+2}$, $P_0(1,-1)$. Find the directional derivative of g at P_0 in the direction of $\bar{u} = 12\hat{i} + 5\hat{j}$.

First, we find the unit vector in the direction of $\bar{u}$.

$$\hat{u} = \frac{\bar{u}}{\|\bar{u}\|} = \frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}$$

length (or norm) of $\bar{u} = \sqrt{12^2+5^2} = 13$

My notation: lower case letters with a bar denote vectors e.g., $\bar{u}$ or $\bar{a}$ ...

lower case letters with a hat $\hat{}$: unit vectors. $\hat{u}$, $\hat{y}$ etc. $\|\hat{u}\| = 1$.

$$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j}$$

$$= \frac{(xy+2)(1-0)-(x-y)(y+0)}{(xy+2)^2}\hat{i} + \frac{(xy+2)(0-1)-(x-y)(x+0)}{(xy+2)^2}\hat{j}$$

$$= \frac{(\cancel{x}y+2) - \cancel{xy} + y^2}{(xy+2)^2}\hat{i} + \frac{-(\cancel{xy}+2) - x^2 + \cancel{xy}}{(xy+2)^2}\hat{j}$$ plug in $P_0(1,-1)$

$$= \frac{y^2+2}{(xy+2)^2}\hat{i} - \frac{(x^2+2)}{(xy+2)^2}\hat{j}$$

$$= \frac{3}{1}\hat{i} + \frac{-3}{1}\hat{j} = 3\hat{i} - 3\hat{j}$$

$$\left(D_{\hat{u}}g\right)_{P_0} = \nabla g \cdot \hat{u} = \left(3\hat{i} - 3\hat{j}\right) \cdot \left(\frac{12}{13}\hat{i} + \frac{5}{13}\hat{j}\right)$$

$$= \frac{3 \times 12}{13} - \frac{3 \times 5}{13} = \frac{21}{13}.$$

Notice the three steps in such problems. First, we find the unit vector $\hat{u}$ in the direction of $\bar{u}$. Then we find the gradient ∇g at the given point P_0. Third, we take the scalar product of ∇g and $\hat{u}$ to get $\left(D_{\bar{u}}g\right)_{P_0}$.