MATH 273 – Lecture 8 (09/18/2014)

Gradient and Directional Derivative

$$D_{\hat{u}} f\Big|_{P_0} = \nabla f\Big|_{P_0} \cdot \hat{u} = |\nabla f|\,\underbrace{|\hat{u}|}_{=1\ (\hat{u}\text{ is unit vector})} \cos\theta = |\nabla f|_{P_0}\Big| \cos\theta$$

where θ is the angle between ∇f and $\hat{u}$.

Using the definition of scalar product, we can infer properties of directional derivative.

Properties of directional derivatives

1. f increases most rapidly when $\theta = 0$, so that $\cos\theta = 1$, i.e., along ∇f itself. The derivative here is $|\nabla f|$.

2. f decreases most rapidly when $\theta = \pi$, so that $\cos\theta = -1$. The derivative here is $-|\nabla f|$.

3. If $\hat{u}$ is orthogonal or perpendicular to ∇f, the directional derivative is zero, as $\theta = \frac{\pi}{2}$, $\cos\theta = 0$. $(D_{\hat{u}} f) = 0$.

Hence the gradient gives the direction of fastest increase (among all possible directions). Going opposite to the gradient sees the fastest decrease in the function.

Prob 21 $f(x,y,z) = \frac{x}{y} - yz$. $P_0(4,1,1)$.

Find the directions in which f increases and decreases most rapidly at P_0. Then find the derivatives in these directions.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{1}{y} - 0\right)\hat{i} + \left(\frac{-x}{y^2} - z\right)\hat{j} + (0 - y)\hat{k}$$

$$= \frac{1}{y}\hat{i} - \left(\frac{x}{y^2} + z\right)\hat{j} - y\hat{k}$$

At $P_0(4,1,1)$, $\nabla f_{P_0} = \frac{1}{1}\hat{i} - \left(\frac{4}{1^2} + 1\right)\hat{j} - 1\hat{k} = \hat{i} - 5\hat{j} - \hat{k}$.

Direction of largest increase $\hat{u} = \frac{\nabla f}{\|\nabla f\|}$ (unit vector along ∇f).

i.e., $\hat{u} = \frac{1}{\sqrt{27}}(\hat{i} - 5\hat{j} - \hat{k})$.

Notice that $\|\nabla f\| = \sqrt{(1)^2 + (-5)^2 + (-1)^2} = \sqrt{27} = 3\sqrt{3}$.

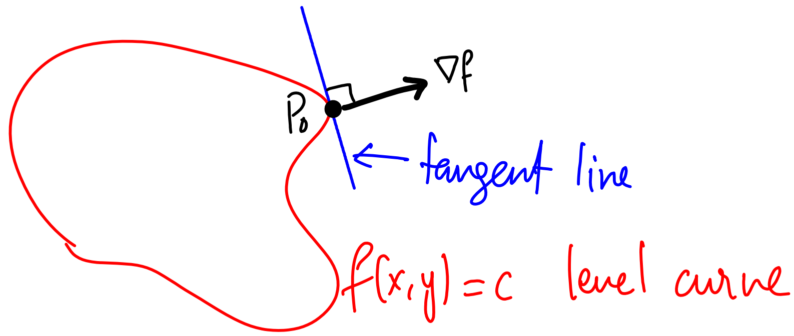
Direction of largest decrease $\hat{v} = -\hat{u} = \frac{-1}{\sqrt{27}}(\hat{i} - 5\hat{j} - \hat{k})$.

$(D_{\hat{u}}f)_{P_0} = |\nabla f|$ (as $(D_{\hat{u}}f) = |\nabla f| \cos 0$) $= \sqrt{27}$. ← largest derivative in any direction

Similarly, $(D_{\hat{v}}f)_{P_0} = -|\nabla f| = -\sqrt{27}$. ← largest decrease, i.e., most negative derivative

Gradients and Tangents Level Curves

Intuition

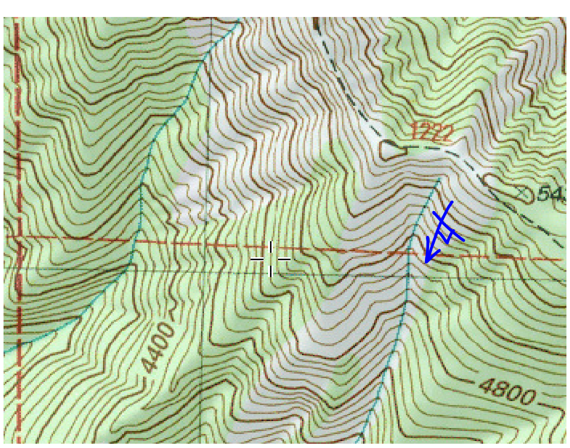


∇f is always normal (or orthogonal) to the level curves.

The tangent at any point on $f(x,y)=c$ gives the direction in which the function is not changing at that point (ie., instantaneously). Since the gradient is perpendicular to the direction in which the derivative is zero, it is perpendicular to the tangent. By finding the gradient, we can find the equation to the tangent line.

We could also prove this result formally – see the textbook.

Illustration: rivers on topo maps are always perpendicular to the contour lines, which are level curves of elevation.



(from gmaps)

On topographical maps, the contour lines are level curves of elevation. Rivers and streams, indicated by blue lines/curves always flow perpendicular to the contour lines, as water seeks the most direct path to lower elevation.

We can write down the equation of the tangent line at point P_0 on the level curve $f(x,y)=c$ by finding the gradient of f at P_0. The slope of the tangent line is $-\dfrac{1}{\text{slope of } (\nabla f)_{P_0}}$. $\longrightarrow$ Recall that the slope of the line perpendicular to a given line with slope m_1 is $-\frac{1}{m_1}$.

<u>Prob 27</u> $\overset{f(x,y)=c}{xy=-4}$, $P_0(2,-2)$. Sketch the level curve $f(x,y)=c$ and ∇f at P_0. Then write the equation for the tangent to the curve at P_0.

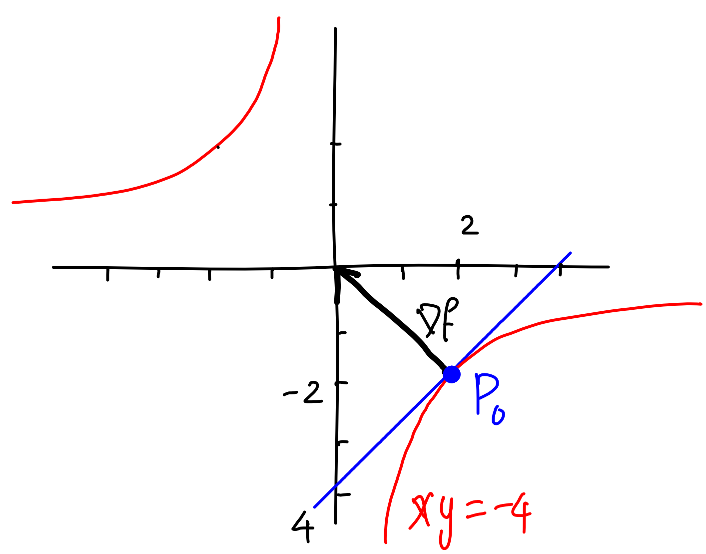
$xy=-4$

i.e., $y=\dfrac{-4}{x}$

$\nabla f = \dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j}$

$= y\hat{i} + x\hat{j}$

at $P_0(2,-2)$, $(\nabla f)_{P_0} = -2\hat{i} + 2\hat{j}$.



Slope of the tangent line at P_0 will be $\dfrac{-1}{\text{Slope of } \nabla f}$.

Slope of $\nabla f = \dfrac{2}{-2} = -1$. Hence, slope of tangent line $= \dfrac{-1}{-1} = 1$.

Hence the tangent line, which passes through $P_0(2,-2)$, is

$$(y - -2) = 1(x-2), \quad \text{i.e.,} \quad y = x-4.$$

The equation of the line with slope m and passing through the point (x_0, y_0) is $(y-y_0) = m(x-x_0)$.

Prob 29 (d) $f(x,y) = x^2 - xy + y^2 - y$. Find direction $\hat{u}$ along which $D_{\hat{u}} f(1,-1) = 4$.

This is a "reverse" problem – you know how to find the directional derivative of the function along a given direction. Here, you are given the directional derivative and are asked to find the direction.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = (2x - y + 0 - 0)\hat{i} + (0 - x + 2y - 1)\hat{j}$$

$$(\nabla f)_{(1,-1)} = 3\hat{i} - 4\hat{j}$$

Let $\hat{u} = u_1\hat{i} + u_2\hat{j}$. $\|\hat{u}\| = 1$ so $u_1^2 + u_2^2 = 1$ —— (1).

We specify the direction as a unit vector. We start with this unknown vector as $\hat{u} = u_1\hat{i} + u_2\hat{j}$, and solve for the two unknowns u_1 and u_2. We need two equations.

Sine $(D_{\hat{u}}f)_{(1,-1)} = 4$, we get

$$(\nabla f)_{(1,-1)} \cdot \hat{u} = 4$$

$$(3\hat{i} - 4\hat{j}) \cdot (u_1\hat{i} + u_2\hat{j}) = 4$$

$$3u_1 - 4u_2 = 4 \quad \text{——} \quad (2)$$

(2) gives $3u_1 = 4u_2 + 4$, So $u_1 = \frac{4}{3}(u_2 + 1)$.

Plugging into (1) gives

$$\left(\frac{4}{3}\right)^2 (u_2+1)^2 + u_2^2 = 1$$

$$16(u_2^2 + 2u_2 + 1) + 9u_2^2 = 9$$

$$25u_2^2 + 32u_2 + 7 = 0$$

notice that $25 + 7 = 32$, which points to the factorization desired

$$(25u_2 + 7)(u_2 + 1) = 0$$

i.e., $u_2 = -\frac{7}{25}$, $u_2 = -1$.

The corresponding values of u_1 are (given by $u_1 = \frac{4}{3}(u_2+1)$)

$$u_1 = \frac{4}{3}\left(\frac{-7}{25}+1\right) \quad \text{and} \quad u_1 = \frac{4}{3}(-1+1).$$

Hence $u_1 = \frac{4}{3}\left(\frac{18}{25}\right) = \frac{24}{25}$ and $u_1 = 0$.

Thus, there are two directions $\hat{u}$ along which $(D_{\hat{u}}f)_{P_0} = 4$. They are $\hat{u} = 0\hat{i} + (-1)\hat{j}$, i.e., $\hat{u} = -\hat{j}$, and

$$\hat{u} = \frac{24}{25}\hat{i} - \frac{7}{25}\hat{j}.$$