MATH 273 – Lecture 9 (09/23/2014)

Prob 35 (Section 13.5)

$$\left(D_{\hat{u}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \hat{u}$$

∇f → gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Setting: You're given $\left(D_{\bar{u}}f\right)_{P_0}$ and $\left(D_{\bar{v}}f\right)_{P_0}$, and asked to find $\left(D_{\bar{w}}f\right)_{P_0}$ for directions $\bar{u}$, $\bar{v}$, and $\bar{w}$ (not necessarily unit vectors). Form of f is not given.

$(D_{\bar{u}}f)$ at $P_0(1,2)$ (x, y) in direction of $\bar{u} = \hat{i} + \hat{j}$ is $2\sqrt{2}$ and in direction of $\bar{v} = -2\hat{j}$ is -3. Find $(D_{\bar{w}}f)_{P_0}$ in the direction of $\bar{w} = -\hat{i} - 2\hat{j}$.

$$\left(D_{\bar{w}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \hat{w} = (\nabla f)_{P_0} \cdot \frac{\bar{w}}{\|\bar{w}\|}$$

Let $(\nabla f)_{P_0} = (f_x)_{P_0}\hat{i} + (f_y)_{P_0}\hat{j} = f_1\hat{i} + f_2\hat{j}$ where f_1 and f_2 are unknown.

$$\left(D_{\bar{u}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \frac{\bar{u}}{\|\bar{u}\|} = (f_1\hat{i} + f_2\hat{j}) \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}} = 2\sqrt{2}$$

i.e., $\sqrt{2}\left(\frac{f_1}{\sqrt{2}} + \frac{f_2}{\sqrt{2}} = 2\sqrt{2}\right)$, which gives $f_1 + f_2 = 4$ —(1)

$$\left(D_{\bar{v}}f\right)_{P_0} = (\nabla f)_{P_0} \cdot \frac{\bar{v}}{\|\bar{v}\|} = (f_1\hat{i} + f_2\hat{j}) \cdot \frac{-2\hat{j}}{2} = \underbrace{\frac{f_2 \cdot -2}{2}}_{\left(f_1 \times 0 + f_2 \times \frac{-2}{2}\right)} = -3$$

i.e., $f_2 = 3$ —— (2). So, (1) gives $f_1 = 1$.

$$(\nabla f)_{P_0} = 1\hat{i} + 3\hat{j} = \hat{i} + 3\hat{j}.$$

$$\bar{w} = -\hat{i} - 2\hat{j}$$

$$\|\bar{w}\| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

Hence $$(D_{\bar{w}} f)_{P_0} = (\nabla f)_{P_0} \cdot \frac{\bar{w}}{\|\bar{w}\|}$$

$$= (\hat{i} + 3\hat{j}) \cdot \frac{(-\hat{i} - 2\hat{j})}{\sqrt{5}} = \left(1 \times \frac{-1}{\sqrt{5}} + 3 \times \frac{-2}{\sqrt{5}}\right) = \frac{-7}{\sqrt{5}}.$$

Tangent Planes and Differentials (Section 13.6)

Extend idea of tangent line of a level curve to that of the tangent plane of a level surface.

Let $\bar{r}(t) = \underbrace{x(t)}_{x}\,\hat{i} + \underbrace{y(t)}_{y}\,\hat{j} + \underbrace{z(t)}_{z}\,\hat{k}$ be a smooth curve on a level surface $f(x,y,z) = c$.

We write $f(x(t), y(t), z(t)) = c$; and apply chain rule (w.r.t t).

$$\frac{df}{dt} = \frac{dc}{dt} = 0$$

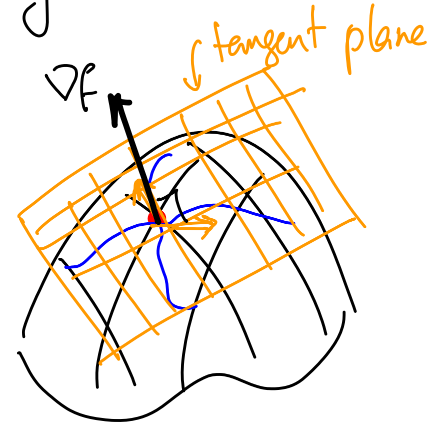
$$\underbrace{\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}} = 0$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)}_{\frac{d\bar{r}}{dt}} = 0$$

$\frac{d\bar{r}}{dt} \longrightarrow$ velocity vector

Hence ∇f is orthogonal to the velocity vector.

All tangent lines to the surface (i.e., tangents to all level curves on the surface) at given point P_0 lie on a plane that is orthogonal to ∇f at P_0.



Def The **tangent plane** at P_0 on the level surface $f(x,y,z)=c$ of a differentiable function f is the plane through P_0 orthogonal to $(\nabla f)_{P_0}$.

The line parallel to $(\nabla f)_{P_0}$ passing through P_0 is the **normal line** of the surface at P_0.

Section 11.5 : Equation of plane perpendicular to $A\hat{i}+B\hat{j}+C\hat{k}$ at $P_0(x_0,y_0,z_0)$ is $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

Equation of a line through P_0 parallel to $\bar{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ is $\bar{r}(t)=\bar{r}_0+t\bar{v}$ for $-\infty<t<\infty$, i.e.,

$$\bar{r}(t)=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix}+t\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix},\quad \text{i.e.,}\quad \begin{aligned}x&=x_0+tv_1\\y&=y_0+tv_2\\z&=z_0+tv_3\end{aligned}$$

Hence, the tangent plane to $f(x,y,z)=c$ at $P_0(x_0,y_0,z_0)$ is

$$\left(\frac{\partial f}{\partial x}\right)_{P_0}(x-x_0)+\left(\frac{\partial f}{\partial y}\right)_{P_0}(y-y_0)+\left(\frac{\partial f}{\partial z}\right)_{P_0}(z-z_0)=0.$$

The normal line is given by

$$x=x_0+\left(\frac{\partial f}{\partial x}\right)_{P_0}t,\quad y=y_0+\left(\frac{\partial f}{\partial y}\right)_{P_0}t,\quad z=z_0+\left(\frac{\partial f}{\partial z}\right)_{P_0}t,\quad -\infty<t<\infty.$$

<u>Prob 2</u> Find equations of (a) tangent plane and (b) normal line at $P_0(3,5,-4)$ to the surface $\underbrace{x^2+y^2-z^2=18}_{f(x,y,z)=c}$.

$$\nabla f=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}$$

$$=2x\hat{i}+2y\hat{j}-2z\hat{k}$$

at $P_0(\underset{x_0,y_0,z_0}{3,5,-4})$, $\nabla f=6\hat{i}+10\hat{j}+8\hat{k}$.

So, the tangent plane is $6(x-3)+10(y-5)+8(z+4)=0$

i.e., $\frac{1}{2}\left(6x+10y+8z+(-18-50+32)=0\right)$

i.e., $3x+5y+4z=18$.

Normal line is $x=3+6t$,

$y=5+10t$

$z=-4+8t$, $\quad -\infty<t<\infty$.