Principles of Optimization (Fall 2024)
Practice Final Examination

- There are **six** problems in this exam, presented in pages 2–3.
- The total points (given in parentheses) add to 100.
- This is a **LIMITED open resources** exam. You **are allowed** to use **class resources** such as lecture notes, homework and exam solutions, handouts, and AMPL files—anything that is **available on the course web page**. You can also **use AMPL**, if needed.
- You are **NOT PERMITTED** to
  - use AI engines such as ChatGPT, CoPilot, etc.,
  - search the internet for the exam (except the course web page),
  - communicate with anyone else on the exam,
  - post on online forums about the exam, or
  - use any *other* textbook, handouts, or online resources.
- You **must start your exam** by writing down the following statement word-by-word, and signing under the same.

  > I promise that I will not seek external help from any person, AI, or the internet while working on this exam. I will use only the resources posted on the class web page while working on the same.
  > —*Signature*

- You **must end your exam** by writing down the following **second** statement word-by-word, and again signing under the same.

  > As promised, I did not use any help from another person or through online or email communications or from AI engines while working on this exam.
  > —*Signature*

- You **must email your submission as a SINGLE PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name should identify you** in the usual manner. If you are Ned Gerblanski, you should name your submission NedGerblanski_Final.pdf (and **NOT** Ned_Gerblanski*.pdf or EdGerblanski*.pdf or ...). You could add anything more to your filename *after* these terms, e.g., NedGerblanski_Final_Math364.pdf. Please avoid white spaces in the file name :-).
- **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, e.g., "NedGerblanski Final Exam submission".**
- This exam must be emailed to me **by 10:00 PM on Thursday, December 12, 2024**.

1. (12) Consider the following LP.

$$\begin{array}{lrcl} \min & z = 4x_1 + \ x_2 + 5x_3 & & \\ \text{s.t.} & 7x_1 + 2x_2 + 2x_3 & \leq & 12 \\ & 3x_1 + \ x_2 - \ x_3 & \geq & 6 \\ & x_1, \ x_2, \ x_3 & \geq & 0 \end{array}$$

   After introducing the variables $s_1, e_2, a_2,$ (slack, excess, and artificial variables for the constraints) and solving the LP, it was found that $\{s_1, x_2\}$ were the basic variables in the optimal tableau in that order. Compute directly (without doing the usual simplex pivots) the optimal tableau for this LP.

2. Consider the following LP and its optimal tableau.

$$\begin{array}{lrcl} \max & z = \ 3x_1 + x_2 & & \\ \text{s.t.} & 3x_1 + 4x_2 & \geq & 6 \\ & 2x_1 + \ x_2 & \leq & 6 \\ & x_1 + 2x_2 & = & 5 \\ & x_1, x_2 & \geq & 0 \end{array}$$

z	x_1	x_2	e_1	s_2	a_1	a_3	rhs
1	a	b	c	$5/3$	M	$M - 1/3$	d
0	0	e	f	$2/3$	g	$5/3$	h
0	i	j	k	$2/3$	l	$-1/3$	m
0	n	1	p	$-1/3$	q	$2/3$	r

   (a) (12) Find the values of the seventeen unknowns $a, b, c, d, e, f, g, h, i, j, k, l, m, n$ and p. You should **not** do all the pivot steps of the big-M method from scratch to identify the unknowns.

   (b) (6) Find the dual of the given LP and its optimal solution from the optimal primal tableau.

   (c) (6) Find the range of values of the right-hand side of the *second* constraint for which the current basis remains optimal.

   (d) (4) Find the new optimal solution and the new optimal objective function value if the right-hand side of the *third* constraint is 6.

3. (18) You are given the following seven 3-letter words: `AFT`, `FAR`, `JOE`, `KEN`, `MOB`, `TAD`, and `ZAP`. We assign a numerical score for each letter of the alphabet, starting with $\text{score}(\texttt{A}) = 1$, $\text{score}(\texttt{B}) = 2$, and so on up to $\text{score}(\texttt{Z}) = 26$. The score of a word then is the sum of the scores of its letters, e.g., $\text{score}(\texttt{KEN}) = 11 + 5 + 14 = 30$. You have to select four of the seven words such that the sum of the scores of the selected words is maximum. Further, the selected words must satisfy the following restrictions.

   - If `ZAP` is selected, then `FAR` cannot be selected.
   - If a word containing the letter `E` is selected, then all other words containing the same letter must also be selected.
   - The four selected words must satisfy

     (sum of letter 1 scores) $<$ (sum of letter 2 scores) $<$ (sum of letter 3 scores).

   Formulate an IP to solve this problem. For your convenience, the scores of each letter in the alphabet is tabulated below.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

4. (15) Consider the following two LPs.

$$\begin{array}{lrcl} \max\ z = & c_1x_1 + c_2x_2 & & \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 & \le & b_1 \\ & a_{21}x_1 + a_{22}x_2 & \le & b_2 \\ & x_1,\ x_2 & \ge & 0 \end{array} \qquad \text{(LP-1)} \qquad \begin{array}{lrcl} \max\ z = & c_1x_1 + 3c_2x_2 & & \\ \text{s.t.} & 10a_{11}x_1 + 30a_{12}x_2 & \le & 20b_1 \\ & a_{21}x_1 + 30a_{22}x_2 & \le & 2b_2 \\ & x_1,\ x_2 & \ge & 0 \end{array} \qquad \text{(LP-2)}$$

   The optimal basis for both LPs is $\{x_1, s_2\}$, and the optimal basis for both corresponding dual LPs is $\{y_1, e_2\}$. (s_2 is the slack variable for the second constraint, y_1 the dual variable for the first constraint, and e_2 the excess variable for the second constraint in the dual LPs.) The optimal solutions for LP-1 and its dual have $x_1 = \alpha, s_2 = \beta, y_1 = 10\gamma, e_2 = \delta$, and $z^* = \theta$. Find the optimal solutions to LP-2 and its dual, and its z^* value in terms of $\alpha, \beta, \gamma, \delta$, and θ.

5. (15) Model the following statement using extra binary variables and big-M.

   if $|2x + 5y| > 2$ **then** $|3x + 4y| \ge 5$.

6. (12) Use complementary slackness conditions and the dual theorem to find the optimal solution to the following LP:

$$\begin{array}{lrcl} \min & z = 3x_1 + 3x_2 + 4x_3 & & \\ \text{s.t.} & 4x_1 + 6x_2 + 3x_3 & \ge & 7 \\ & 3x_1 + x_2 + x_3 & \ge & 3 \\ & x_1,\ x_2,\ x_3 & \ge & 0 \end{array}$$