Principles of Optimization (Fall 2024)
Simplex Method in Matrix Form, and Sensitivity Analysis

Note: Most of the material discussed in this handout will also be covered in the lecture notes.

1 Matrix Form of Simplex Method

Consider the standard form max-LP given in matrix-form:

$$\begin{array}{lrcl} \max & z = \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0}. \end{array} \tag{1}$$

Notice that $\mathbf{c}^T$ represents the transpose of $\mathbf{c}$. The necessary slack and surplus/artificial variables have already been added (and hence we have all constraints in the $=$ form). Here, A is an $m \times n$ matrix (i.e., m rows and n columns), $\mathbf{b}$ is an m-vector, $\mathbf{c}$ and $\mathbf{x}$ are n-vectors. Thus, the n variables (represented as $\mathbf{x}$) includes the slack/surplus/artificial variables.

Suppose we know which of the n variables are *basic* in the optimal tableau. Since there are m constraints, there will be m basic variables and $(n-m)$ non-basic variables. Grouping all the basic variables together (and all the non-basic variables together), we split the vector $\mathbf{x}$ into $\begin{bmatrix}\mathbf{x}_B \\ \mathbf{x}_N\end{bmatrix}$, where $\mathbf{x}_B$ denotes the m basic variables, and $\mathbf{x}_N$ the non-basic variables. Subsequently, we also group the columns of A corresponding to the variables $\mathbf{x}_B$ into the $m \times m$ matrix B, and the remaining non-basic columns into the $m \times (n-m)$ matrix N. The cost vector is similarly divided as $\mathbf{c}^T = [\mathbf{c}_B^T \; \mathbf{c}_N^T]$. Thus, the starting tableau for the simplex method can be written as follows.

z	$\mathbf{x}_B$	$\mathbf{x}_N$	rhs
$\mathbf{1}$	$-\mathbf{c}_B^T$	$-\mathbf{c}_N^T$	0
$\mathbf{0}$	B	N	$\mathbf{b}$

(2)

Notice here that $\mathbf{0}$ is the m-vector of all zeros. Applying the simplex method, the optimal tableau looks something like what is shown below:

z	$\mathbf{x}_B$	$\mathbf{x}_N$	rhs
$\mathbf{1}$	$\mathbf{0}$	$?$	$?$
$\mathbf{0}$	I	$?$	$?$

(3)

We know that the basic variables form the canonical columns in the optimal tableau, thus giving the identity matrix I as shown. The important task of course is to figure out what each of the $?$'s are. Recall that the steps of the simplex method are equivalent to those of the Gauss-Jordan method (we are performing elementary row operations in both cases). Hence, we can think of obtaining the optimal tableau by multiplying the initial tableau by a suitable matrix on the left (which is the *elementary* or *transformation* matrix representing all the ERO's). Let us call this transformation matrix as T (notice that it will be an $(m+1)\times(m+1)$ matrix). If you concentrate on the columns of z and $\mathbf{x}_B$, you can see that multiplying by T converts this part of the tableau to the identity matrix:

$$T \cdot \begin{bmatrix} 1 & -\mathbf{c}_B^T \\ \mathbf{0} & B \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}. \tag{4}$$

Hence, we must have that

$$T = \left(\begin{bmatrix} 1 & -\mathbf{c}_B^T \\ \mathbf{0} & B \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1 & \mathbf{c}_B^T B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix}. \tag{5}$$

You should check that the inverse of T is indeed as given here. The multiplication looks similar to the multiplication of 2×2 matrices, except that the elements themselves are matrices or vectors. Knowing T, we can write the optimal tableau as follows.

$$\begin{bmatrix} 1 & \mathbf{c}_B^T B^{-1} \\ \mathbf{0} & B^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\mathbf{c}_B^T & -\mathbf{c}_N^T & 0 \\ \mathbf{0} & B & N & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} & -\mathbf{c}_N^T + \mathbf{c}_B^T B^{-1} N & \mathbf{c}_B^T B^{-1} \mathbf{b} \\ \mathbf{0} & I & B^{-1} N & B^{-1} \mathbf{b} \end{bmatrix}. \tag{6}$$

Thus, the optimal objective function value is given by $z^* = \mathbf{c}_B^T B^{-1} \mathbf{b}$, and the optimal solution is given by $\mathbf{x}_B = B^{-1}\mathbf{b}$, $\mathbf{x}_N = \mathbf{0}$.

We will use the Farmer Jones example to illustrate the various cases of sensitivity analysis that we want to study. To make the example interesting, the objective function coefficient of acres of wheat (x_2) will be set at 25 (as opposed to the original value of 100—think about the price per bushel of wheat being set at \$1 in place of \$4). The LP is given below.

$$\begin{array}{llcll} \max & z = 30x_1 + 25x_2 & & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq & 7 & \text{(total acres)} \\ & 4x_1 + 10x_2 & \leq & 40 & \text{(labor hrs)} \\ & x_1 & \geq & 3 & \text{(min corn)} \\ & x_1, \quad x_2 & \geq & 0 & \text{(non-neg)} \end{array}$$

Notice that the (min corn) constraint has been divided by 10 throughout. The starting tableau and the optimal tableau for the above LP are the following.

Table 1: Starting tableau

z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
1	-30	-25	0	0	0	M	0
0	1	1	1	0	0	0	7
0	4	10	0	1	0	0	40
0	1	0	0	0	-1	1	3

Table 2: optimal tableau

z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
1	0	5	30	0	0	M	210
0	0	1	1	0	1	-1	4
0	0	6	-4	1	0	0	12
0	1	1	1	0	0	0	7

The variables that are basic in the optimal tableau are $\mathbf{x}_B^T = [e_3 \; s_2 \; x_1]$, with the optimal solution given by $x_1 = 7$, $s_2 = 12$, $e_3 = 4$, and $z^* = 210$. Hence, $\mathbf{c}_B^T = [0 \; 0 \; 30]$, and the matrix B and its inverse are as given below.

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Check the inverse of B. In fact, we can read of B^{-1} from the optimal tableau! Note that the element in the right-bottom position of T is B^{-1}. Hence, if some of the columns in the initial tableau had the identity matrix I in the rows 1 to m, then the same columns will have B^{-1} in the optimal tableau. For example, if the non-basic part of A were in fact identity, i.e., $N = I$, then in the optimal tableau, we will have (under the columns of $\mathbf{x}_N$) $B^{-1}N = B^{-1}I = B^{-1}$.

At the same time, the slack and artificial variables indeed form the identity matrix in the initial tableau (before converting it to canonical form) – see the columns of s_1, s_2, a_3 in the initial tableau (Table 1). Hence, B^{-1} can be read off from the columns of $s_1, s_2,$ and a_3 from the optimal tableau (check to make sure that this observation indeed is true).

2 Sensitivity Analysis

We are now ready to consider the various cases of sensitivity analysis. The optimal tableau will remain optimal as long as

1. $-\mathbf{c}_N^T + \mathbf{c}_B^T B^{-1} N \geq 0$ (for a max-LP, all entries in the z-row should be non-negative for optimality), and
2. $B^{-1}\mathbf{b} \geq 0$ (in order to maintain feasibility).

We will use these two conditions to check whether the current basis still remains optimal after one (or more) of the parameters is (are) changed.

2.1 Changing c_j when x_j is non-basic

Here, x_j is one of the variables in $\mathbf{x}_N$. We denote the column of N corresponding to x_j by $\mathbf{a}_j$. Consider changing c_j to $c_j + \Delta$. From Equation (6), we can see that the only entry in the optimal tableau that possibly changes is the x_j-entry in Row-0. As seen from the optimal tableau for the Farmer Jones problem, x_2 is a non-basic variable. Consider changing the objective function coefficient of x_2 from 25 to $25 + \Delta$ (you could consider the yield per acre of wheat changing from 25 to $25 + \Delta$). Our goal is to find the range of values of Δ for which the current basis still remains optimal. The new entry for x_2 in Row-0 of the optimal tableau is

$$\tilde{c}_2 = -(25 + \Delta) + \mathbf{c}_B^T B^{-1}\mathbf{a}_2 = (-25 - \Delta) + \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix}$$

$$= (-25 - \Delta) + \begin{bmatrix} 30 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 10 \\ 0 \end{bmatrix} = 5 - \Delta.$$

Notice that $\mathbf{a}_2$ is the column of the matrix A corresponding to x_2. We need $\tilde{c}_2 \geq 0$ for the current basis to remain optimal (Condition (1)). Hence, the range of values of Δ for which the current basis remains optimal is $\Delta \leq 5$. Notice that the optimal solution and the optimal objective function value do not change when Δ is in this range. Also, recall that the optimal solution was to farm corn in all the 7 acres available ($x_1 = 7, x_2 = 0$ in the optimal solution). With $\Delta \leq 5$, the revenue per acre of wheat is ≤ 30 (which is the value for corn). At the same time, each acre of wheat requires more hours of labor (10) than an acre of corn (4). There is no minimum level of wheat production required either. Hence, it makes sense not to farm any wheat. If the revenue per acre of wheat becomes higher than that of corn (> 30), one should farm some acres of wheat in order to make the maximum revenue.

This value, 5, is called the *reduced cost* of the non-basic variable x_2. The reduced cost of a non-basic variable (in the optimal tableau) of a max-LP is defined as the maximum amount by which its objective function coefficient can increase while the current basis still remains optimal. Hence, if the objective function coefficient of wheat changes to 32 (i.e., $\Delta = 7$), the current basis is no longer optimal – in fact, it is *sub-optimal* in the sense that the value of x_2 can be increased from 0 to some positive value and the value of z^* can be increased. In this case, in order to find the new optimal tableau, one will have to perform one more simplex pivot (with x_2 as the entering variable).

2.2 Changing c_j when x_j is basic

Here, x_j is one of the variables in $\mathbf{x}_B$. Consider changing the objective function coefficient of x_1 from 30 to $30 + \Delta$. $\mathbf{c}_B^T$ also changes here – $\mathbf{c}_B^T = \begin{bmatrix} 0 & 0 & 30+\Delta \end{bmatrix}$. Since $\mathbf{c}_B$ changes, all the entries in Row-0 corresponding to $\mathbf{x}_N$ (i.e., all the non-basic variables) could possibly change. The new entries in Row-0 under $\mathbf{x}_N$ are given by

$$\tilde{\mathbf{c}}_N^T = -\mathbf{c}_N^T + \mathbf{c}_B^T B^{-1} N = \begin{bmatrix} -25 & 0 & M \end{bmatrix} + \begin{bmatrix} 0 & 0 & 30+\Delta \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -25 & 0 & M \end{bmatrix} + \begin{bmatrix} 30+\Delta & 30+\Delta & 0 \end{bmatrix} = \begin{bmatrix} 5+\Delta & 30+\Delta & M \end{bmatrix}.$$

Using the Condition (1) again, the current basis remains optimal if all entries in $\tilde{\mathbf{c}}_N$ are ≥ 0, i.e., if $5 + \Delta \geq 0$ and $30 + \Delta \geq 0$. Hence, the range of values of Δ for which the current basis remains optimal is $\Delta \geq -5$. For this range, the revenue per acre of corn will be at least as big as the value for wheat (25), and hence one will not grow any wheat. If $\Delta = -8$ for example, then each acre of wheat will yield a higher revenue than an acre of corn (25 and 22 respectively). The current basis becomes sub-optimal in this case. In order to find the new optimal solution, one will have to pivot x_2 into the basis (perform one more simplex pivot).

2.3 Changing the right-hand-side (b_i) of the i-th constraint

Consider changing b_i to $b_i + \Delta$. From Equation (6), we can see that the entries in the rhs (last) column will change. For example, consider changing the rhs of acres constraint to $7 + \Delta$ ($i = 1$, i.e., the first constraint here). This change models the situation where Farmer Jones has some extra land to farm. The new rhs vector can be written as

$$\tilde{\mathbf{b}} = \begin{bmatrix} 7+\Delta \\ 40 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 40 \\ 3 \end{bmatrix} + \begin{bmatrix} \Delta \\ 0 \\ 0 \end{bmatrix}. \text{ Hence,}$$

$$B^{-1}\tilde{\mathbf{b}} = B^{-1}\mathbf{b} + \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \begin{bmatrix} \Delta \\ -4\Delta \\ \Delta \end{bmatrix} = \begin{bmatrix} 4+\Delta \\ 12-4\Delta \\ 7+\Delta \end{bmatrix} = \tilde{\mathbf{x}}_B. \tag{7}$$

Using the Condition (2), in order for the current basis to still remains optimal, we need all elements of $\tilde{\mathbf{x}}_B \geq 0$. In other words, $4 + \Delta \geq 0,\ 12 - 4\Delta \geq 0$, and $7 + \Delta \geq 0$, which give $\Delta \geq -4,\ \Delta \leq 12/4 = 3,\ \Delta \geq -7$, or $-4 \leq \Delta \leq 3$. The current basis remains optimal as long as Δ is in this range. Notice that, when $\Delta > 3$, Jones will have more than 10 acres to farm, but could only use 10 due to the limit on the labor hours (why?). Thus the optimal solution will change when $\Delta = 4$, for instance. Similarly, when $\Delta = -4.5$, the minimum level of corn cannot be satisfied (as Jones will have only 2.5 acres), and hence the current basis will no longer be optimal.

For $-4 \leq \Delta \leq 3$, the new optimal objective function is given by (using the expression for $B^{-1}\tilde{\mathbf{b}}$ from Equation (7))

$$\tilde{z}^* = \mathbf{c}_B^T \tilde{\mathbf{x}}_B = \mathbf{c}_B^T B^{-1} \tilde{\mathbf{b}} = \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} 4 + \Delta \\ 12 - 4\Delta \\ 7 + \Delta \end{bmatrix} = 210 + 30\Delta.$$

Hence the shadow price of the “acres” constraint is 30. Recall that the shadow price of a constraint is defined as the *improvement* (increase for a max-LP) in the objective function value for a unit increase the rhs value of the constraint.

2.4 Changing the column (both c_j and $\mathbf{a}_j$) when x_j is non-basic

The general method outlined above can also be used to calculate the new optimal tableau (and optimal basis) when more than one parameter is changed. For example, consider changing the objective function coefficient of x_2 (which is non-basic) from 25 to 35, and at the same time, changing its coefficient in the “labor hours” constraint from 10 to 8 (i.e., the number of labor hours required for an acre of wheat is now 8). Using the formulas given in Equation (6), the new entries in the column of x_2 are given by

$$\begin{bmatrix} -c_2 + \mathbf{c}_B^T B^{-1} \mathbf{a}_2 \\ \hline B^{-1}\mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} -35 + \begin{bmatrix} 30 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -5 \\ \hline 1 \\ 4 \\ 1 \end{bmatrix}.$$

Notice that we used the result for $\mathbf{c}_B^T B^{-1} = \begin{bmatrix} 30 & 0 & 0 \end{bmatrix}$, which we already saw in section 2.1. Since the modified entry in the Row-0 under x_2 is no longer non-negative (it is -5), the current basis is no longer optimal. Intuitively, it required less number of labor hours and generates more revenue to farm an acre of wheat after these changes. In order to find the new optimal tableau, the column of x_2 in the original optimal tableau should be replaced with the one derived above, and one more simplex pivot should be performed. The new optimal solution thus obtained is $e_3 = 1, x_2 = 4, x_1 = 3$, with $\tilde{z}^* = 225$.