Principles of Optimization (Spring 2024): Homework 1

- The total points (given in parentheses) add up to 105. You will be graded for 100 points (with the possibility of getting up to 5 points as extra credit).
- **This homework is due at the start of class on Thursday, Aug 29.**
- **Alternatively, you can submit your homework by email as follows:**
  - **You must email your submission as a PDF file to kbala@wsu.edu**. You are welcome to write answers by hand, and scan the writings (or take pictures of your writings) into a **PDF file**.
  - **Your file name should identify you in this manner: If you are Beatrice McGullicutty, say, you should name your submission BeatriceMcGullicutty_Math364_Hw1.pdf. Please avoid white spaces in the file name (use "_" or "-" instead).**
  - **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, expression, e.g., "BeatriceMcGullicutty Math364 Hw1 submission".**
  - **This homework is due by 12:05 PM on Thursday, August 29, i.e., email me before start of the lecture.**

1. (20) Recall that an $n \times n$ matrix A is symmetric if $A^T = A$, or equivalently, $A_{ij} = A_{ji}$ for all i, j.

   (a) If $B = A + A^T$, show that B is symmetric.

   (b) If $B = AA^T$, show that B is symmetric.

   Notice that A need not be symmetric in the above two cases. You must give the argument for *general* values of n; in particular, it is not enough to show the result for $n = 2$ or $n = 3$.

2. (15) Recall that for a square matrix $A \in \mathbb{R}^{n \times n}$, $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$, i.e., the sum of its diagonal entries. Let A and B be matrices such that both AB and BA are defined. Show that

   $$\text{trace}(AB) = \text{trace}(BA).$$

   Notice that A and B need *not* be square here. Again, provide a general argument, and not just for the cases of $n = 2$ or $n = 3$.

3. (10) Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$. Is the set of vectors $V = \{\mathbf{v}_!, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent or dependent. Why?

4. (25) Let B be an invertible matrix. Describe the inverse of B' in each of the following cases, which are modifications of B, in terms of modifications to B^{-1}.

   (a) $B' = 100B$.

   (b) B' is obtained by multiplying every entry in Row 1 of B by 2.

   (c) B' is obtained by multiplying every entry in Column 1 of B by 2.

5. (20) Show that $(A^{-1})^T = (A^T)^{-1}$.

6. (15) Let $A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$. What conditions must a, b, c, d satisfy so that A is invertible? Assuming these conditions hold, find A^{-1}.