Principles of Optimization (Fall 2024): Homework 9

- There are **two problems** with multiple subproblems for each, and the total points (given in parentheses) add up to 115. You will be graded for 110 points (with the possibility of getting up to 5 points as extra credit).
- **You must submit your homework by email as follows:**
  - **You must email your submission as a PDF file to kbala@wsu.edu**. You are welcome to write answers by hand, and scan the writings (or take pictures of your writings) into a **PDF file**.
  - **Your file name should identify you in this manner: If you are Tuong Lu Kim, say, you should name your submission TuongKim_Math364_Hw9.pdf. Please avoid white spaces in the file name (use "_" or "-" instead).**
  - **Begin the SUBJECT of your email submission with the same `FirstnameLastname` expression, e.g., "TuongKim Math364 Hw9 submission".**
- **This homework is due by 5:00 PM on Friday, November 8.**

1. (35) Consider the following LP:

$$\begin{array}{lrcl} \max\ z = & x_1 + 2x_2 + 3x_3 & & \\ \text{s.t.} & -2x_1 + x_2 + x_3 & \leq & 2 \\ & x_2 + 2x_3 & \leq & 4 \\ & 2x_1 + x_2 + x_3 & \leq & 3 \\ & x_1,\ x_2,\ x_3 & \geq & 0 \end{array}$$

One can verify that $\begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/4 & 0 & 1/4 \\ 1 & -1 & 1 \\ -1/2 & 1 & -1/2 \end{bmatrix}$.

(a) Show that the solution $\mathbf{x}^*$ with x_1, x_2, x_3 as the basic variables is optimal. You must **not** solve the LP using tableau simplex method here. What is the optimal solution $\mathbf{x}^*$?

(b) Let the right-hand side (rhs) vector of the constraints be changed to $k\mathbf{b}$ for $k \geq 0$, where $\mathbf{b} = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}^T$ is the current rhs vector. What is the new optimal solution in terms of $\mathbf{x}^*$, the original optimal solution?

(c) Let the rhs vector be changed now to $\ell\mathbf{b}$ for $\ell < 0$. Can you find the new optimal solution quickly based on your previous responses? Why or why not?

2. (80) Woeful Co. makes three kinds of waffles named $W1, W2$, and $W3$. The two main ingredients in all three waffles are egg and sugar. Based on the data for each waffle type in terms of how much egg and sugar (in ounces) are required as well as the unit profit (in \$) for one waffle, and the total availability of egg and sugar (also in ounces), Woeful writes down the following LP to determine its production plan, with x_i being the number of type-i waffles made, for $i = 1, 2, 3$:

$$\begin{array}{lrcrl} \max\ z = & 3x_1 + 2x_2 + 3x_3 & & & \text{(total profit)} \\ \text{s.t.} & 3x_1 + 2x_2 + x_3 & \leq & 400 & \text{(egg limit)} \\ & 2x_1 + 3x_2 + 2x_3 & \leq & 600 & \text{(sugar limit)} \\ & x_1,\ x_2,\ x_3 & \geq & 0 & \text{(non-neg)} \end{array}$$

   (a) Consultant Prof. Woebegone tells Woeful that the optimal plan is to make only waffles $W1$ and $W3$. Given this info, write down the optimal tableau of the above LP, after adding slack variables s_1, s_2 for the egg and sugar constraints, respectively. What is the optimal solution (including optimal z^*)? You must **not** use the tableau simplex method to get the optimal tableau here.

   (b) For what values of the unit profit for $W2$ (originally set at \$2) does the current basis remain optimal? What would be the new optimal solution (and the new total profit) for Woeful's LP if the unit profit for one $W2$ waffle is \$5?

   (c) For what values of the unit profit for $W1$ (originally set at \$3) does the current basis remain optimal? What would be the new optimal solution (and the new total profit) for Woeful's LP if the unit profit for one $W1$ waffle is \$7?

   (d) For what amounts of total egg available (originally at 400 oz) will the current basis remain optimal? Find the shadow price of the (egg limit) constraint.

   (e) For what amounts of total sugar available (originally at 600 oz) will the current basis remain optimal? Find the shadow price of the (sugar limit) constraint.

   (f) Woeful is considering a fourth type of waffle $W4$, which uses 2 ounces each of egg and sugar per waffle, and yields a profit of \$4 per waffle. Should Woeful make any of $W4$ waffles? Why? Again, you must **not** re-solve the original LP from scratch after adding an extra variable x_4 to answer this question.