MATH 364: Lecture 1 (08/20/2024)

This is Principles of Optimization.

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In WSU since 2004, and have been in USA since 1999. I'm originally from India. If you do not understand what I say because of my accent, do let me know ☺!

My research interests are in optimization, algebraic topology, applications to biology, medicine, etc.

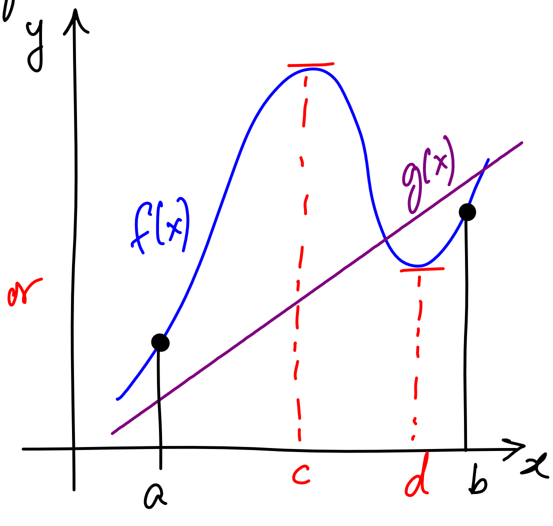
Optimization — what is it?

In Calculus, you would've seen problems of the form min/max $f(x)$ for $a \le x \le b$.

$f'(x)=0$ gives critical points. In addition,

$$\left.\begin{array}{l} f''(x) > 0 \text{ gives minima} \\ f''(x) < 0 \text{ gives maxima} \end{array}\right\}$$

these are local maxima or minima!



But we have to examine the **end points** of the interval as well!

Here, the minimum in the interval $[a,b]$ is at $x=a$, an end point.

If $g(x)$ is linear, the maxima/minima are at the end points!

In Math 364, we extend this easier linear case to higher dimensions.

If $g(x_1, x_2, \dots, x_n)$ is linear, the optima still occur at corner points ($\equiv$ end points).

A Motivating Problem

"Math"

Dude M. Major has a Thursday Problem.

Has 5 hrs, $48 to spare

	Costs	Utility
— can get tutoring	$8/hr	2/hr
— Can party	$16/hr	3/hr

How many hours to get tutored, and how many to party so that total utility is maximized?

Two decisions: $\begin{cases} x_1 = \#\text{ hrs to get tutored} \\ x_2 = \#\text{ hrs to party} \end{cases}$

objective/goal: maximize total utility while not exceeding the total hours and cash available.

"maximize"

"subject to"

$$\begin{aligned} \max \quad & 2x_1 + 3x_2 & & \text{(total utility)} \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 & & \text{(time available)} \\ & 8x_1 + 16x_2 \leq 48 & & \text{(cash available)} \\ & x_1, x_2 \geqslant 0 & & \text{(non-negativity)} \end{aligned}$$

linear optimization model/problem or linear program (LP).

We will go through LP formulation problems of this kind in detail.

If Dude could spend all time and money, we can write

$$x_1 + x_2 = 5$$
$$8x_1 + 16x_2 = 48$$

Ignore utility for now — it may well be not ideal to spend all money and time from a utility point-of-view. But we will come back to it later.

This is a system of linear equations of the form $A\bar{x} = \bar{b}$

vectors ($\bar{a}, \bar{b}, \bar{x}, \bar{0}$, etc.) → Bala's notation!

How do you solve $A\bar{x}=\bar{b}$? → Should've learned all about it in Math 220!

- ∗ form $[A|\bar{b}]$, the augmented matrix;
- ∗ use elementary row ops (EROs) to reduce $[A|\bar{b}]$ to echelon form and then to reduced row echelon form (RREF).

notation:

R_i : i^{th} row

EROs:

1. Exchange/swap row $R_i \rightleftarrows R_j$
2. Scaling: multiply R_i by $\alpha \neq 0$ αR_i
3. Replacement: $R_i \leftarrow R_i + \alpha R_j$ (or just $R_i + \alpha R_j$)
   replace row i by the sum of itself and α times row j ($\alpha \neq 0$ for nontrivial ERO).

$$\left[\begin{array}{cc|c} \textcircled{1} & 1 & 5 \\ 8 & 16 & 48 \end{array}\right] \xrightarrow{R_2 - 8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 8 & 8 \end{array}\right] \xrightarrow{R_2(\frac{1}{8})} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & \textcircled{1} & 1 \end{array}\right] \xrightarrow{R_1 - R_2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$x_1 = 4$, $x_2 = 1$ is the unique solution

↳ Dude has to study for 4 hrs and party for only 1 hr

!

This process of taking $[A|\bar{b}]$ to echelon form, and then to RREF is called Gaussian elimination, or the Gauss-Jordan method.

Basic and Non-basic variables

Assume A is $m\times n$ $(m\leq n)$ now (i.e., more general, not square).

$$[A|\bar{b}] \xrightarrow{\text{EROs}} [\tilde{A}|\tilde{\bar{b}}] \longrightarrow \text{RREF}$$

After applying Gaussian elimination on $[A|\bar{b}]$ to get $[\tilde{A}|\tilde{\bar{b}}]$, the variables that have a coefficient of 1 in one row and zero everywhere else are called basic variables (BV). All variables that are not basic are non-basic variables (NBV).

The system $A\bar{x}=\bar{b}$

① has no solution if $[\tilde{A}|\tilde{\bar{b}}]$ has a row of the form $[0\,0\cdots 0|\tilde{b}_i\neq 0]$ (system is inconsistent)

② If $[\tilde{A}|\tilde{\bar{b}}]$ has no such inconsistent row then

(a) if all variables are basic, then the system has a unique solution;

(b) if there are free variables, the system has infinitely many solutions.

We had seen an instance of 2(a), giving $x_1=4, x_2=1$ as the unique solution.

Now assume that partying is also \$8/hr. Then we get

$$\left[\begin{array}{cc|c} \textcircled{1} & 1 & 5 \\ 8 & 8 & 48 \end{array}\right] \xrightarrow{R_2-8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & 8 \end{array}\right] \to \text{inconsistent!}$$

This is an example of Case ①.

Moving on, assume the cash is \$40 now. We get

→ non-basic or free variable

$$\left[\begin{array}{cc|c} \textcircled{1} & 1 & 5 \\ 8 & 8 & 40 \end{array}\right] \xrightarrow{R_2-8R_1} \left[\begin{array}{cc|c} \textcircled{1} & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

This system is an example of 2(b): infinitely many solutions.

Since we have many solutions, we could try to find one that maximizes total utility $(2x_1+3x_2)$. Then Dude will party for all 5 hrs!

Once Dude insisted that he has to use up all the time and money, the objective (of maximizing total utility) did not play any role in finding the solution. The system has a unique solution in this case $(x_1=4,\ x_2=1)$.

But in the last case, where there are infinitely many solutions, the objective will play a part. In this case, as long as $x_1+x_2=5$, and $x_1, x_2 \geq 0$, the solution is valid. Among all such solutions, we could pick the one that gives the largest value for $2x_1+3x_2$ (total utility). Hence $x_2=5$ (and $x_1=0$) gives the optimal solution here, i.e., Dude should just party all five hours!

We will learn (later) that these non-basic variables, which are also called free variables, are critical for solving linear optimization problems.

MATH 364: Lecture 2 (08/22/2024)

Today: * Linear algebra review
  - matrix transpose, rank, inverse

  * Gauss-Jordan method in general

Example for Case 2(b) (for $A\bar{x}=\bar{b}$ with infinitely many solutions)

Consider the following system:
$$\left\{\begin{array}{l} x_1 + 2x_2 + 2x_3 = 6 \\ 3x_1 + 6x_2 + 5x_3 = 8 \end{array}\right\} \qquad m=2,\ n=3$$

Since there are $n=3$ variables, and only $m=2$ equations here, we will have at least one free variable.

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 2 & 6 \\ 3 & 6 & 5 & 8 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 6 \\ 0 & 0 & -1 & -10 \end{array}\right] \xrightarrow[\text{then } (-1)R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & -14 \\ 0 & 0 & \textcircled{1} & 10 \end{array}\right]$$

$$\begin{array}{rl} x_1 + 2x_2 & = -14 \\ x_3 & = 10 \end{array}$$

x_1, x_3 are basic

x_2 is free or non-basic

$$\underbrace{\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -14 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s,}_{\text{parametric vector form of all solutions}} \quad s \in \mathbb{R}$$

set of all real numbers

We can choose x_2 as any real value s, and for each choice, we get a (different) solution for the original system.

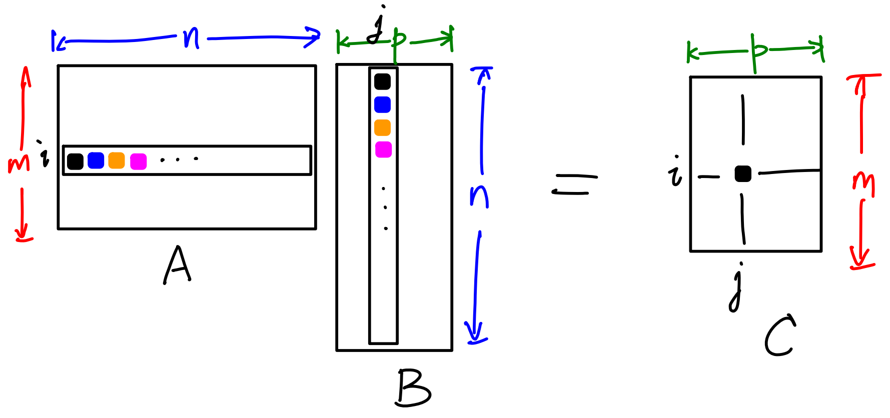
Transpose of a matrix $A \in \mathbb{R}^{m \times n}$

If $B = A^{\overset{\text{transpose}}{T}}$ then $B_{ij} = A_{ji}$ interchange rows and columns.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & -1 & 4 \end{bmatrix}_{2 \times 3} \qquad A^{T} = \begin{bmatrix} 2 & 5 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}_{3 \times 2}$$

Matrix Multiplication

If $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, then $C = AB$ is in $\mathbb{R}^{m \times p}$.



$C_{ij} = ?$

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} + \cdots + A_{in}B_{nj} = \sum_{k=1}^{n} A_{ik}B_{kj}.$$

Rules of matrix multiplication

- $\underbrace{AB \neq BA}_{\to \text{ not symmetric}}$ typically (BA might not even be defined)
- $(AB)C = A(BC) \longrightarrow$ is associative
- $(AB)^{T} = B^{T}A^{T}$

⋮ (several more)

Linear Independence (LI) of vectors

set of m-vectors with real entries

Let $V = \{\bar{v}_1, \dots, \bar{v}_n\}$, where $\bar{v}_j \in \mathbb{R}^m$

"Definition"

Def A **linear combination** of vectors in V is a vector

$$\bar{u} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n, \text{ where } c_j \in \mathbb{R}\ \forall j.$$

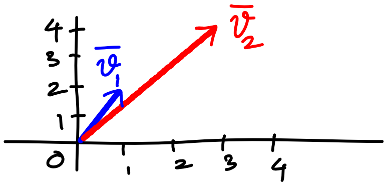
"for all"

If $c_j = 0$ for all j, $\bar{u}$ is the zero vector. This is the **trivial linear combination** of the vectors in V.

Def The vectors in V are **linearly independent** (LI) if the only linear combination of those vectors that is equal to the zero vector is the trivial linear combination.

e.g., $\bar{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix}3\\4\end{bmatrix}$.

$\bar{v}_1$ and $\bar{v}_2$ are not along the same line



If $c_1\bar{v}_1 + c_2\bar{v}_2 = \begin{bmatrix}0\\0\end{bmatrix}$, what are c_1, c_2?

Solve for c_1, c_2 (as a system of linear equations):

$$\left[\begin{array}{cc|c} \textcircled{1} & 3 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow[\substack{\text{then} \\ R_2(-\frac{1}{2})}]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

The unique solution is $c_1 = c_2 = 0$. Hence $\{\bar{v}_1, \bar{v}_2\}$ is LI.

Def If there is a nontrivial linear combination of $\bar{v}_j$'s that is the zero vector, then V is **linearly dependent** (LD).

e.g., $\bar{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix}-3\\-6\end{bmatrix}$, then $3\bar{v}_1 + \bar{v}_3 = \begin{bmatrix}0\\0\end{bmatrix}$, showing $\{\bar{v}_1, \bar{v}_2\}$ is LD.

Note: If $\overset{\swarrow \text{ zero vector}}{\bar{0}} \in V$, then V is LD.

Say $\bar{v}_1 = \bar{0}$. Then $c_1\bar{v}_1 + 0\bar{v}_2 + 0\bar{v}_3 + \cdots + 0\bar{v}_n = \bar{0}$ for any $c_1 \neq 0$ is a non-trivial linear combination that is the zero vector.

Rank of a matrix

Def The **rank** of $A \in \mathbb{R}^{m\times n}$ is the size of a largest LI subset of its rows or its columns.

Def $\text{rank}(A) = \#$ pivot columns in echelon form of A.

Examples

$A = \begin{bmatrix}1 & 2 & 0\\2 & 4 & 1\end{bmatrix}$; $\text{rank}(A) = 2$. e.g., $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is an LI subset of columns

$C = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ $\text{rank}(C) = 1$. $\left(\begin{bmatrix}2\\4\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix}\right)$.

$O = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ $\text{rank}(O) = 0$, as $\begin{bmatrix}0\\0\end{bmatrix}$ is LD by itself,

↓ big "Oh"

Because $c_1\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ for any $c_1 \neq 0$.

↙ Also, we noted above that any set that contains $\bar{0}$ is LD.

How to tell if $V=\{\bar{v}_1,\dots,\bar{v}_n\}$, $\bar{v}_j\in\mathbb{R}^m$, is LI?

- $*$ If $n>m$, V is LD → more vectors than # entries in each of them $\Rightarrow$ LD.
- $*$ If $n\leq m$, then form $A=[\bar{v}_1\,\bar{v}_2\cdots\bar{v}_n]$ ($m\times n$ matrix) and find $\operatorname{rank}(A)$ (= # pivot columns in echelon form of A)
  - if $\operatorname{rank}(A)<n$ then V is LD
  - if $\operatorname{rank}(A)=n$ then V is LI.

e.g., $V=\{\bar{v}_1,\bar{v}_2,\bar{v}_3\}=\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}2\\0\\4\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$. Is V LI?

$$A=\begin{bmatrix}\textcircled{1}&2&3\\0&0&1\\2&4&0\end{bmatrix}\xrightarrow{R_3-2R_1}\begin{bmatrix}1&2&3\\0&0&\textcircled{1}\\0&0&-6\end{bmatrix}\xrightarrow{R_3+6R_2}\begin{bmatrix}\textcircled{1}&2&3\\0&0&\textcircled{1}\\0&0&0\end{bmatrix}$$

$\operatorname{rank}(A)=2$, So V is LD.

Notice that one need not go to the reduced row echelon form of A to identify the number of pivot columns — echelon form will do. In simpler words, rank (A) = # pivot columns in A.

Inverse of a matrix

square matrix

Def For $A \in \mathbb{R}^{m\times m}$, if there is another matrix $B \in \mathbb{R}^{m\times m}$ such that $AB = BA = I_m$, then B is the **Inverse** of A.

m×m identity matrix

We denote this fact by $B = A^{-1}$. Similarly, $A = B^{-1}$.

Here, we say that A is **invertible**.

Why study inverses?

For $A\bar{x} = \bar{b}$ with $A \in \mathbb{R}^{m\times m}$ and invertible, we can do

$$A^{-1}(A\bar{x} = \bar{b})$$

multiply by A^{-1} on the left (on both sides)

"implies"

$$\Longrightarrow (A^{-1}A)\bar{x} = A^{-1}\bar{b}$$

$$\Longrightarrow I\bar{x} = A^{-1}\bar{b} \quad \text{or} \quad \bar{x} = A^{-1}\bar{b}$$

Thus, knowing A^{-1} we can solve $A\bar{x} = \bar{b}$ directly.

How to invert $A \in \mathbb{R}^{m\times m}$? Use GJ!

If $[A \mid I_m] \xrightarrow{\text{EROs}} [I_m \mid B]$, then $B = A^{-1}$.

But if we do not get I_m in place of A, then A is not invertible.

e.g., $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ \boxed{1} & 3 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \rightleftarrows R_2]{R_1-2R_2 \text{ then}} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & \boxed{-1} & 1 & -2 \end{array}\right] \xrightarrow[R_1-3R_2]{R_2\times(-1) \text{ then}} \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right| \left.\underbrace{\begin{matrix} 3 & -5 \\ -1 & 2 \end{matrix}}_{B}\right]$$

$B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is A^{-1}. Check: $AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. ✓

Can invert 2x2 matrices directly using formula:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\underbrace{ad-bc}_{\text{determinant}} \neq 0$, then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

here $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $2\times3-1\times5 = 1 \neq 0$, so $A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Here we are solving two systems $A\bar{x} = \bar{b}_1$ and $A\bar{x} = \bar{b}_2$ for the same A matrix simultaneously. More generally, for $A \in \mathbb{R}^{m\times m}$, we solve m systems of the form $A\bar{x} = \bar{e}_j$, $j = 1, .., m$, where $\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th}$ position is the j^{th} unit vector (or the j^{th} column of the identity matrix I_m).

Gauss-Jordan (GJ) Method in general

$A \in \mathbb{R}^{m \times n}$, $\bar{b} \in \mathbb{R}^m$.

$$[A|\bar{b}] \xrightarrow{\text{EROs}} \begin{array}{c} \\ \updownarrow r \\ \updownarrow m-r \end{array} \begin{array}{c} \begin{array}{ccc} \quad\quad & \leftarrow (n-r) \rightarrow & \quad \end{array} \\ \left[\begin{array}{cc|c} I_r & \tilde{N} & \tilde{\bar{b}}_1 \\ O & O & \tilde{\bar{b}}_2 \end{array}\right] \end{array}$$

zero matrices

Here, $\operatorname{rank}(A) = r$.

1. If $\tilde{\bar{b}}_2 \neq \bar{0}$ (at least one entry is nonzero), then the system is inconsistent.
2. If $\tilde{\bar{b}}_2 = \bar{0}$, we can ignore the last $(m-r)$ rows of zeros.

Assume the variables are split such that

$\bar{x} = \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix}$, where $\bar{x}_B$ are the r basic variables and $\bar{x}_N$ are the $n-r$ non-basic variables.

$\left[\begin{array}{cc|c} I_r & \tilde{N} & \tilde{\bar{b}}_1 \end{array}\right]$ gives

$$I_r \bar{x}_B + \tilde{N} \bar{x}_N = \tilde{\bar{b}}_1$$

$$\Rightarrow \quad \bar{x}_B = \tilde{\bar{b}}_1 - \tilde{N} \bar{x}_N$$

→ free vars!

If we set $\bar{x}_N = \bar{z}$ ($n-r$ vector of parameters), this is the parametric vector form!

MATH 364: Lecture 3 (08/27/2024)

Today:
- * GJ method, example
- * hints on HW1 problems
- * LP formulations

The Relevant case of Gauss-Jordan Method

We consider the case when $\text{rank}(A) = m = \#$ rows of A, i.e., when none of the equations are redundant. We get

$$[A \,|\, \bar{b}] \xrightarrow{GJ} [I_m \;\; \tilde{N} \,|\, \tilde{\bar{b}}]$$

We do not get the zero matrices at the bottom. Also, $\tilde{\bar{b}}_1 = \tilde{\bar{b}}$.

We can split A into $[\overset{\leftarrow m \rightarrow}{B} \; \overset{\leftarrow\rightarrow}{N}]$ $(n-m)$, where B are all the pivot columns.

$\text{rank}(B) = m$, so B is invertible. So B^{-1} exists.

$A\bar{x} = \bar{b}$ is equivalent to $[B \; N]\begin{bmatrix}\bar{x}_B \\ \bar{x}_N\end{bmatrix} = \bar{b}$, i.e.,

$$B^{-1}\left(B\bar{x}_B + N\bar{x}_N = \bar{b}\right)$$

$$\Rightarrow \underbrace{B^{-1}B}_{I_m}\bar{x}_B + B^{-1}N\bar{x}_N = B^{-1}\bar{b}$$

$$\Rightarrow \bar{x}_B = B^{-1}\bar{b} - B^{-1}N\bar{x}_N$$

$$= \tilde{\bar{b}}_1 - \tilde{N}\bar{x}_N.$$

Again, this is the parametric vector form of the solutions.

Example from Lecture 2:

$$\left\{\begin{array}{l} x_1+2x_2+2x_3=6 \\ 3x_1+6x_2+5x_3=8 \end{array}\right\} \qquad \bar{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-14\\0\\10\end{bmatrix}+\begin{bmatrix}-2\\1\\0\end{bmatrix}s, \quad s\in\mathbb{R}$$

basic variables

Let us take $BV=\{x_1,x_3\}$, $NBV=\{x_2\}$ (as given to us).
Then we can split A as follows into $[B\ N]$.

$$A=\left[\begin{array}{ccc} x_1 & x_2 & x_3 \\ 1 & 2 & 2 \\ 3 & 6 & 5 \end{array}\right], \quad \bar{b}=\begin{bmatrix}6\\8\end{bmatrix} \Rightarrow B=\begin{bmatrix}1&2\\3&5\end{bmatrix},\ N=\begin{bmatrix}2\\6\end{bmatrix}, \text{and } \bar{x}_B=\begin{bmatrix}x_1\\x_3\end{bmatrix}, \bar{x}_N=[x_2].$$

B

N

recall, for $B=\begin{bmatrix}a&b\\c&d\end{bmatrix}$,

$$B^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$$

$$B^{-1}=\frac{1}{(1\times5-3\times2)}\begin{bmatrix}5&-2\\-3&1\end{bmatrix}=\begin{bmatrix}-5&2\\3&-1\end{bmatrix}.$$

$$\Rightarrow \tilde{b}=B^{-1}\bar{b}=\begin{bmatrix}-5&2\\3&-1\end{bmatrix}\begin{bmatrix}6\\8\end{bmatrix}=\begin{bmatrix}-14\\10\end{bmatrix}, \quad \tilde{N}=B^{-1}N=\begin{bmatrix}-5&2\\3&-1\end{bmatrix}\begin{bmatrix}2\\6\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$$\Rightarrow \bar{x}_B=\begin{bmatrix}x_1\\x_3\end{bmatrix}=\tilde{b}-\tilde{N}s=\begin{bmatrix}-14\\10\end{bmatrix}-\begin{bmatrix}2\\0\end{bmatrix}s, \quad s\in\mathbb{R}.$$

Combining with $x_2=s$ (free variable), we can write

$$\bar{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-14\\0\\10\end{bmatrix}+\begin{bmatrix}-2\\1\\0\end{bmatrix}s, \quad s\in\mathbb{R},$$

which is what we got originally.

Problems from Homework 1

1. (a) Show $B = A + A^T$ is symmetric

B is symmetric if $B^T = B$.

$$B^T = (A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B.$$

follow from properties of matrix transpose

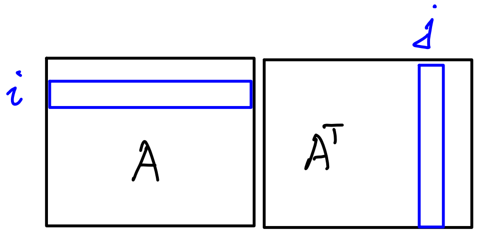
OR

Show $B_{ij} = B_{ji}$ (for B to be symmetric)

$$B_{ij} = A_{ij} + [A^T]_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = B_{ji}.$$

(b) $B = AA^T$

$B_{ij} \overset{?}{=}$



$$B_{ij} = \sum_{k=1}^{n} A_{ik} \cdot A^T_{kj} = \sum_{k=1}^{n} A_{ik} A_{jk} = \sum_{k=1}^{n} A_{jk} A_{ik} = B_{ji}$$

OR

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B.$$

since $(AB)^T = B^T A^T$ in general.

You could try on small instances, e.g., 2x2, 2x3, etc, to identify the pattern or rule, but you must present general arguments as above.

4. Recall: to find inverse of B, we apply GJ to

$$[B|I] \xrightarrow{\text{EROs}} [I|B^{-1}]$$

(b) $B \xrightarrow{2R_1} B'$.

$$[B'|I] \xrightarrow{\text{EROs}} [I|\overset{\text{in terms of } B^{-1}\ ?}{?}]$$

Can start with $[B|I] \xrightarrow{2R_1} [B'|\overset{\text{almost } I}{I'}]$

Try to argue how B^{-1} changes if we started with the additional ERO $(2R_1)$.

<u>Another approach: Elementary matrices</u>

$B \xrightarrow{2R_1} B'$ means $B' = EB$, where $E = \overset{\text{elementary matrix}}{\begin{bmatrix} 2 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}}$.

$I \xrightarrow{2R_1} E$ (apply same ERO to identity)

Then use the result on inverse of product of matrices: $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B')^{-1} = (EB)^{-1} = B^{-1}E^{-1}$$

Explain what the effect of multiplying B^{-1} by E^{-1} will be.

$$E^{-1} = \begin{bmatrix} 1/2 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \text{ here.}$$

Linear Optimization Formulations

We study how to create models for optimization problems arising in many different real life situations. The typical scenarios we work with involve minimizing costs or maximizing revenue or profit subject to meeting demands, or meeting limits on available resources. These models are also called linear programming (LP) formulations.

The main defining criterion is for us to be able to write the objective function and constraints as **linear functions** or **(in)equalities** of the variables. We illustrate the process on an example.

(Taken from *Introduction to Mathematical Programming* by Winston and Venkataramanan.)

Farmer Jones must decide how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. Wheat can be sold at $4 per bushel, and corn at $3 per bushel. Seven acres of land and 40 hours of labor per week are available. Government regulations require that at least 30 bushels of corn need to be produced in each week. Formulate and solve an LP which maximizes the total revenue that Farmer Jones makes.

→ we will do this part later

There is no algorithm (or, step-by-step rules to follow) using which one could write every LP. We list some guidelines here. As you become more familiar with such problems, you will be able to do them more directly (rather than follow a step-by-step procedure).

0. Make notes of various numbers mentioned in the problem

This step is highly recommended, at least for the first several LP formulations you write.

	corn	wheat	availability
land	→	→	7 acres
labor hrs	4 hr/wk	10 hr/wk	40 hours
yield	10 bu/acre	25 bu/acre → bushels/acre	
selling price	$3/bu	$4/bu	
restriction	x		30 bu/wk

1. Define the **decision variables** (d.v.'s)

Let x_1 = # acres of corn
x_2 = # acres of wheat

} it is important to declare the d.v.'s explicitly (as done here). Saying "x_1 = corn", for instance, will not work!!

Goal: Express the objective and constraints (restrictions) as linear functions or (in)equalities of these d.v.s.

2. Define the **objective function**

Usually, maximize revenue/profit, minimize cost, etc.

Goal: maximize total revenue here

Notice that the units for each term is \$: (price/bu) x (bu/acre) x (# acres)

maximize $\$3 \cdot 10 \cdot x_1 + \$4 \cdot 25 \cdot x_2$ (total revenue)

price \$/bu   yield #bu/acre   # acres

In short, $\max \; z = \underbrace{30}_{}x_1 + \underbrace{100}_{}x_2$ (total revenue)

"maximize"

objective function coefficients (of x_1, x_2)

convention: we usually denote the objective function as z (more explanation coming later on!)

You **must** include a short explanation in parentheses for the objective function, and for each (set of) constraint(s) you write. Later on, when we introduce the software AMPL, we could use these explanations to denote the constraints and the objective function.

3. Define **constraints**

Constraint 1: land availability

$$x_1 + x_2 \leq 7 \quad \text{(land availability)}$$

acres of corn

acres of wheat

total land available

Constraint 2: labor hrs

$$4x_1 + 10x_2 \leq 40 \quad \text{(labor hrs)}$$

hrs/acre of corn

acres of corn

Constraint 3: Government regulation

$$10x_1 \geq 30 \quad \text{(min. corn)}$$

"at least"

bushels/acre of corn

acres of corn

4. Define **sign restrictions** on variables

x_1, x_2 are # acres, so negative values do not make sense.

$$x_1, x_2 \geq 0 \quad \text{(non-negativity)}.$$

There are scenarios where negative values might make perfect sense. for example, when modeling a budget balance, a deficit could be a negative value, and a surplus is a positive value.

If no explicitly mentioned as non-negative, the variables are assumed to be unrestricted in sign (urs).

Putting it all together, we get the (entire) LP formulation:

$$\begin{array}{llrcll} \max & z = 30x_1 + 100x_2 & & & & \text{(total revenue)} \\ \text{s.t.} & & x_1 + x_2 & \leq & 7 & \text{(land availability)} \\ & & 4x_1 + 10x_2 & \leq & 40 & \text{(labor hrs)} \\ & & 10x_1 & \geq & 30 & \text{(min corn)} \\ & & x_1, x_2 & \geq & 0 & \text{(non-negativity)} \end{array}$$

subject to

MATH 364: Lecture 4 (08/29/2024)

Today:
- alternative formulation for Farmer Jones LP
- assumptions of LP
- graphical solution in 2D

Alternative formulation for Farmer Jones LP

(Taken from *Introduction to Mathematical Programming* by Winston and Venkataramanan.)

Farmer Jones must decide how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. Wheat can be sold at \$4 per bushel, and corn at \$3 per bushel. Seven acres of land and 40 hours of labor per week are available. Government regulations require that at least 30 bushels of corn need to be produced in each week. Formulate and solve an LP which maximizes the total revenue that Farmer Jones makes.

$$\left.\begin{array}{llll} \max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\ & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\ & 10x_1 & \geqslant 30 & \text{(min corn)} \\ & x_1,\ x_2 & \geqslant 0 & \text{(non-negativity)} \end{array}\right\}$$

Formulation with
x_1 = # acres of corn
x_2 = # acres of wheat

We now consider a different set of d.v.'s.

Let $x_i = \underbrace{\text{\# bushels of crop } i,\ i=1,2,\ 1=\text{corn},\ 2=\text{wheat}.}_{\text{such short-hand notation will be handy in many cases!}}$

$$\begin{array}{llll} \max & z = 3x_1 + 4x_2 & & \text{(total revenue)} \\ \text{s.t.} & \underset{\text{acres/bu of corn}}{\left(\frac{1}{10}\right)}x_1 + \underbrace{\left(\frac{1}{25}\right)}_{\text{\# acres of wheat}}x_2 & \leq 7 & \text{(land availability)} \\ & 4\left(\frac{1}{10}\right)x_1 + 10\left(\frac{1}{25}\right)x_2 & \leq 40 & \text{(labor hrs)} \\ & x_1 & \geqslant 30 & \text{(min corn)} \\ & x_1,\ x_2 & \geqslant 0 & \text{(non negativity)} \end{array}$$

Naturally, the two formulations are equivalent — as you will confirm in homework 2.

Assumptions of LP Formulations

1. **Proportionality**: Contribution of a variable to objective function or to the left-hand side of a constraint is proportional to its value.

   This assumption is central to linearity. Hence we get terms of the form $30x_1$, $10x_2$, etc., and not $5x_1^2$ or $\frac{6}{x_j}$.

2. **Additivity**: Total contribution from all variables is the sum of the contributions from individual variables.

   Another assumption central to linearity. Hence we get expressions of the form $30x_1+100x_2$, $4x_1+10x_2$, etc., and not $30x_1/100x_2$, for instance.

3. **Divisibility**: The variables can take fractional values, e.g., can farm corn in 2.8 acres.

   But we might insist on x_j to take only integer values in some cases, e.g., whether to reopen the government, modeled by a binary variable, for which fractional values do not make sense.

   Then we get integer programming (IP), which we will introduce later.

4. **Certainty** All coefficients and right-hand side values are known beforehand with surety.

   In stochastic LP, we assume probability distributions on some of the data (objective function or constraint coefficients, or rhs numbers).

   Another approach is robust optimization, where we want to find solutions that are optimal over ranges of values of the input parameters.

How do we solve LPs? We extend the ideas used to solve systems of linear equations in 2D to solve LPs in 2D, to start with. Then we talk about extending these ideas to higher dimensions.

Graphical method to solve LPs in 2D

We illustrate the procedure on the original formulation of the Farmer Jones LP.

$$\begin{array}{llrl} \max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\ & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\ & 10x_1 & \geqslant 30 & \text{(min. corn)} \\ & x_1, x_2 & \geqslant 0 & \text{(non-negativity)} \end{array}$$

The first step is to plot the feasible region of the LP.

Def The set of all $(x_1, \ldots, x_n)$ satisfying all constraints, including sign restrictions is the **feasible set** of the LP. It is also called the **feasible region**.

We start by plotting the feasible region of the Farmer Jones LP by plotting the constraints. → all constraints are inequalities here, including nonnegativity constraints.

How to plot $x_1 + x_2 \leq 7$?

First, plot $x_1 + x_2 = 7$ (can use $A(7,0)$ and $B(0,7)$ as two points)
↳ need two points to plot a straight line

Then we pick the "correct" side to plot the "≤" constraint, by testing any one point. The obvious choice is to use (0,0). Here, $0+0 \leq 7$, and hence (0,0) is on the correct side. We indicate the correct side by drawing arrow(s).

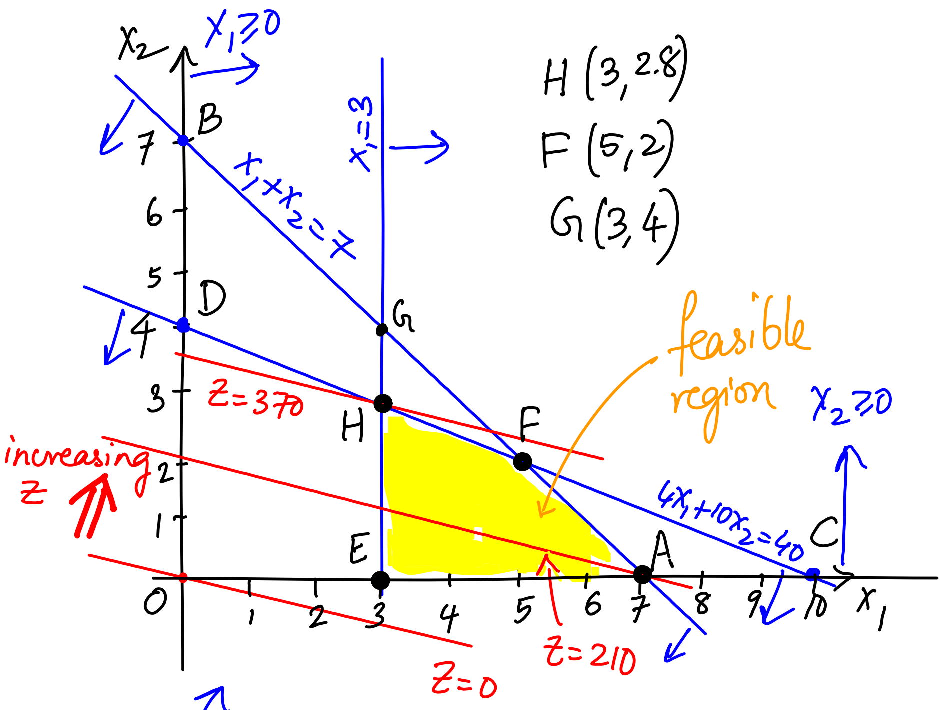
$4x_1 + 10x_2 \le 40$

We plot the "=" line using C(10,0) and D(0,4). Again, (0,0) is on the correct side for the "≤" inequality.

$10x_1 \ge 30$

We need only one point, e.g., E(3,0), as the line $x_1 = 3$ is vertical. Here, (0,0) is on the wrong side of the "≥" inequality.

$x_1, x_2 \ge 0$ } gives the first quadrant



(Such rough figures will do for graphical solutions of LPs!

The region AEHF is the feasible region of the LP, which is the intersection of all the half-spaces.

H: point of intersection of $x_1=3$ and $4x_1+10x_2=40$, i.e., H(3,2.8).

It is critical to solve for these points of intersection, rather than trying to guess them from the rough diagram.

F: $x_1 + x_2 = 7$

$\underline{4x_1 + 10x_2 = 40}$

F(5,2)

H: $x_1 = 3$

$\underline{4x_1 + 10x_2 = 40}$

H(3,2.8)

Note that there are infinitely many points in the feasible region here.

or feasible points

Def A point in the feasible region is a **feasible solution**. Any point not in the feasible region is an **infeasible solution**.

An infeasible solution violates at least one constraint, e.g., G(3,4) is an infeasible point, as it violates $4x_1 + 10x_2 \le 40$.

Def An **optimal solution** is a solution (or point) in the feasible region at which the objective function is optimum, i.e., it is maximum for a max objective function, and minimum for a min objective function.

How to find an optimal solution? → We involve the objective function in the picture now.

We plot a line corresponding to one value of z. Notice that for any value of z, the objective function is a straight line, of the form $30x_1 + 100x_2 = z$.

For $z = 210$, $(7,0)$ and $(0,2.1)$ are two points. → picked 210, as it's a multiple of 30, and hence could find a point on it quickly.

Then we change the value of z, i.e., plot lines parallel to the first line. Since only the rhs value is changing, the slope remains the same, i.e., $\frac{-100}{30}$.
→ right-hand side

A parallel z-line through the origin has $z=0$. Hence the good direction here is to slide the z line up (since we want to maximize z).

Looks like we could slide the z-line up to F, and then to H. Remember that we need to stay within the feasible region, and hence cannot go any further than H.

$F(5,2)$: $z = 30x_1 + 100x_2 = 30\times5 + 100\times2 = 350$

$H(3,2.8)$: $z = 30\times3 + 100\times2.8 = 370$. ✓

} In some cases, it will be obvious which corner point is the optimal solution even from the rough diagram. But in other cases, we need to verify using calculations.

So, $H(3,2.8)$ is the optimal solution.

Interpretation: Jones needs to farm corn in 3 acres and wheat in 2.8 acres, to get a maximum total revenue of \$370/wk.

Note: 3 acres of corn are required to produce ≥ 30 bushels of corn.

He is using $3+2.8 = 5.8$ out of 7 acres, and

$4(3) + 10(2.8) = 40$ labor hours.

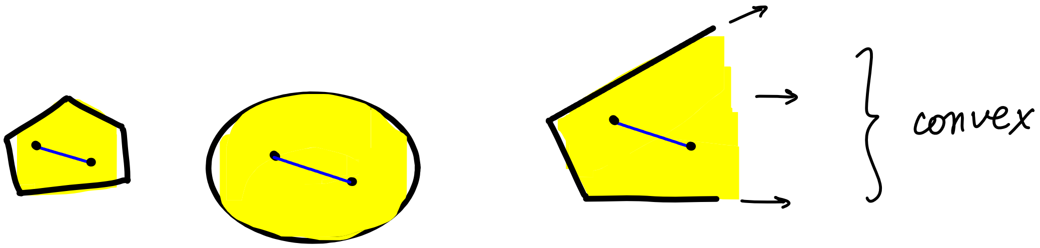
So, the (labor hrs) constraint is satisfied as an equation.

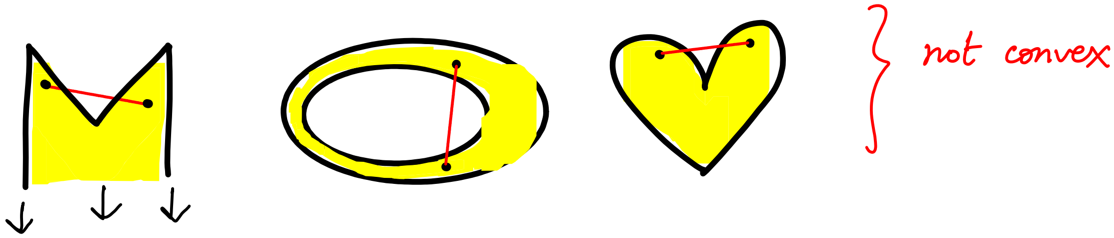
Def A constraint that is satisfied as an equation at the optimal solution is a **binding constraint**. A constraint that is satisfied as a strict inequality at the optimal solution is a **non-binding constraint.**

At H(3, 2.8), the (labor hrs) and (min corn) constraints are binding, while the (land availability) constraint is non-binding.

It turns out that we need to look at only the vertices of the feasible region for the optimal solution. This result follows from the fact that the feasible region of an LP is a **convex set**.

Def A set S is **convex** if the line segment joining any two points in S lies entirely inside S.





MATH 364: Lecture 5 (09/03/2024)

Today: * cases of LP
* one full example

Hw2, Problem 1

Candidate choices for d.v.'s:

1. x_i = # hrs in paint shop for toy i, $i=1,2$, $1 \equiv$ dirty, $2 \equiv$ ugly
2. x_i = # toy i (per day) i, $i=1,2$, $1 \equiv$ dirty, $2 \equiv$ ugly
3. x_{ij} = # hrs of toy i in shop j, $i,j=1,2$; $i=1 \equiv$ dirty, $i=2 \equiv$ ugly
$j=1 \equiv$ assembly,
$j=2 \equiv$ paint

Think proportionality!

1500 Dirty C's per day in assembly shop $\Rightarrow$
$\left(\frac{1}{1500}\right)$ day of assembly is required for each Dirty C.

For assembly shop, can use it for assembling Dirty C's or Ugly C's all day.

(total time for Dirty C's) + (total time for Ugly C's) ≤ 1 (assembly)

1 day

Similar constraint for paint shop.

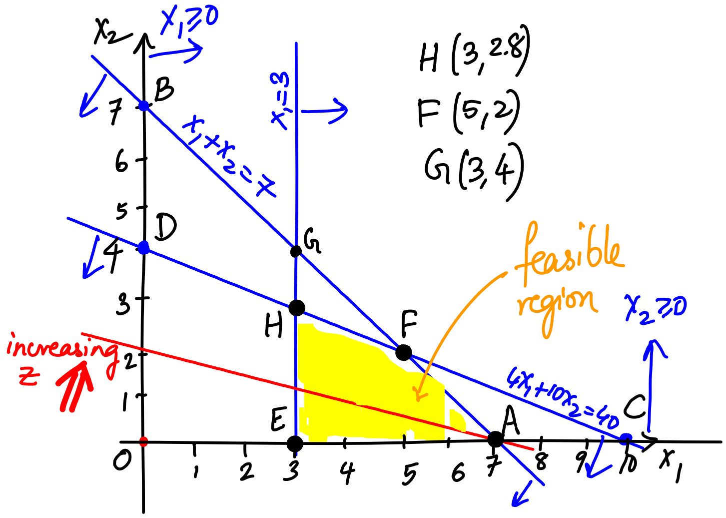
If x_i = # hrs in paint shop for toy i,

figure out the # toys of type i (using proportionality)

Objective function: 4(# Dirty C's) + 3(# Ugly C's) (total profit)

Recall feasible region of Farmer Jones LP is AEHF.

AEHF is a convex region.



Def The corners of the feasible region of an LP are called as **extreme points** or **vertices**.

Because of its convexity and linearity (defined by linear inequalities), if an LP has an optimal solution and its feasible region has extreme points, an optimal solution will occur at an extreme point!

We now consider cases of LP, which correspond to the cases of systems of linear equations. Recall that such a system has a unique solution, infinitely many solutions, or no solutions

Cases of LP

Case 1 **Unique optimal solution**. If the LP has a unique optimal solution, that optimal solution will be a corner point of its feasible region.

e.g., Jones LP. H(3,2.8) is the unique optimal solution.

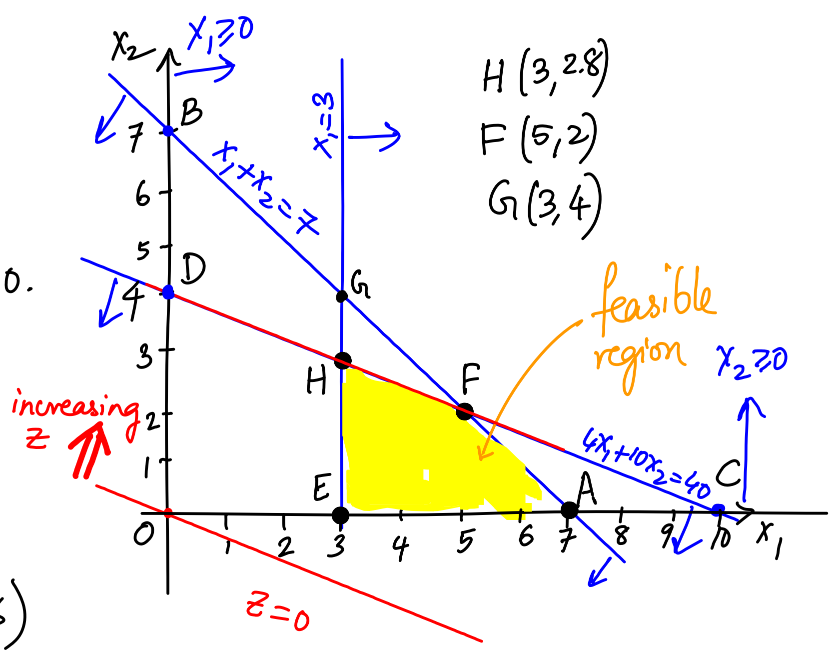
Case 1 is the good, typical case of LPs. But there are three other special cases of LP (Cases 2, 3, and 4).

Case 2 Some LPs have infinitely many optimal solutions, i.e., they have **alternative optimal solutions**.

For example, in the Jones LP, "if price of corn is also \$4/bushel", the revenue function becomes

$$\max \quad z = 40x_1 + 100x_2 \qquad \text{(total revenue)}$$

Note that slope of the z-line is now equal to the slope of the (labor hrs) line $4x_1+10x_2 \leq 40$.



At F(5,2),

$z = 40(5)+100(2) = 400$

We get $z=400$ at $H(3,2.8)$ as well : $40(3)+100(2.8) = 400$.

The same z-value is obtained at every point on $\overline{HF}$.

Note: The slope of z-line must be same as that of a binding constraint for the LP to be Case 2.

Case 3 Some LPs have no feasible solutions, and hence no optimal solutions. Such LPs are called as **infeasible LPs**.

Case 4 **Unbounded LPs**.

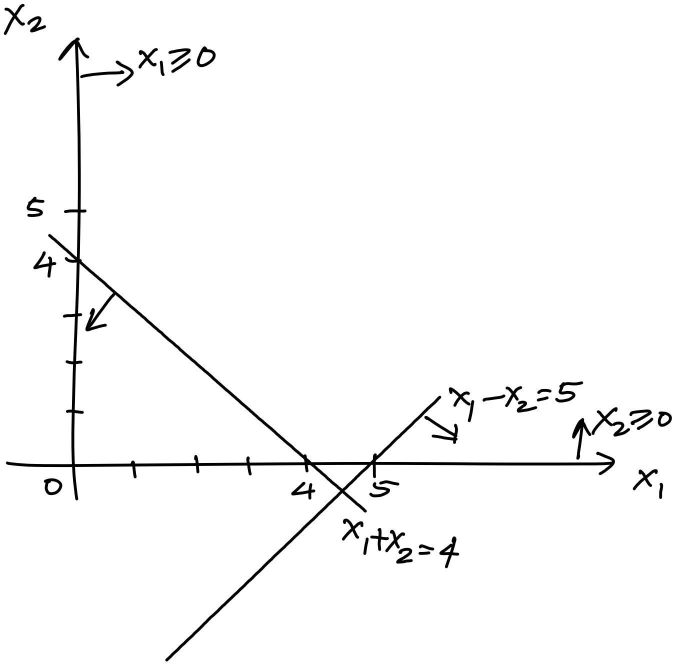
There are feasible solutions with arbitarily large z-values for max LPs or arbitrarily small z-values for min LPs. Hence an unbounded LP has no optimal solutions.

Cases 1, 2, and 3 correspond to the three cases of systems of linear equations ($A\bar{x}=\bar{b}$) — unique optimal solution, infinitely many solutions, and infeasible systems. Case 4 (unbounded LPs) is unique to LPs, i.e., there is no corresponding case in $A\bar{x}=\bar{b}$.

We now consider several example LPs, and identify which case each one belongs to.

LP instances

1. $$\begin{aligned} \max\ & z = x_1 + x_2 \\ \text{s.t.}\ & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$



The feasible region is empty, i.e., it's a Case 3 LP (infeasible LP).

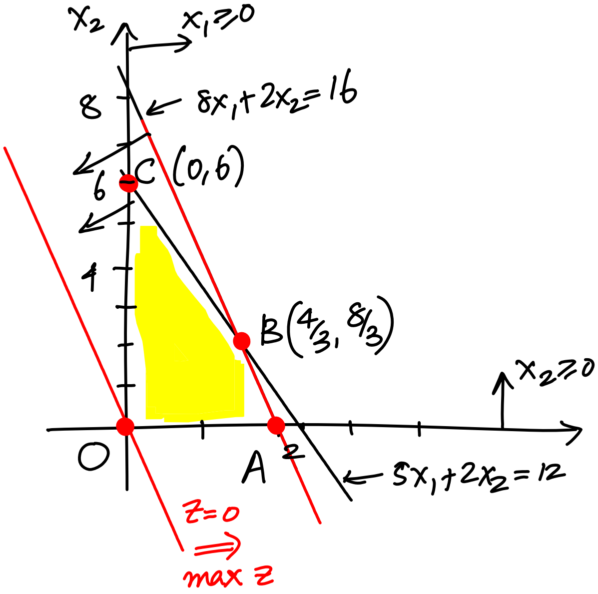
Typically, when we get an infeasible LP in practice, it indicates we do not have enough raw materials to satisfy all demands, for instance.

2. $\max \ z = 4x_1 + x_2$

s.t. $8x_1 + 2x_2 \leq 16$

$5x_1 + 2x_2 \leq 12$

$x_1, x_2 \geq 0$



B: $8x_1 + 2x_2 = 16$

$5x_1 + 2x_2 = 12$

$x_1 = \frac{4}{3}, \ x_2 = 8/3$

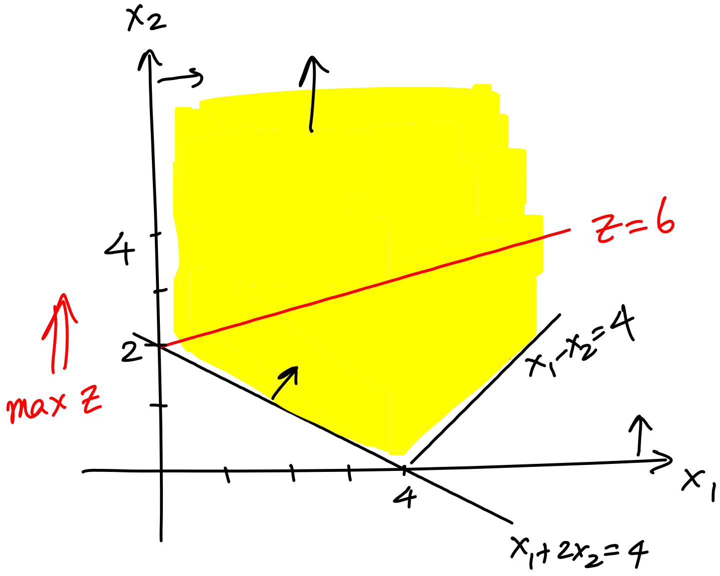
Every point on $\overline{AB}$ is an optimal solution. At each such point, $z = 4(2) + (0) = 8$. This is a Case 2 LP, i.e., it has alternative optimal solutions.

3. $\max \ z = -x_1 + 3x_2$

s.t. $x_1 - x_2 \leq 4$

$x_1 + 2x_2 \geq 4$

$x_1, x_2 \geq 0$



Can slide the z line up without any limit.

Case 4 LP.

A set (or region) is bounded if it can be enclosed in a finite box (can be large). It is unbounded if no such finite box exists.

One Full example

Formulate and solve LP:

Richy Rich trades currencies, and is working with the Crooner, the currency of ImaginationLand, and the US Dollar (USD). He can buy Crooners at the rate of \$0.20 USD per Crooner, and can buy USD at the rate of 3 Crooner per USD. Let x_1 be the number of USD bought by paying Crooners, and x_2 the number of Crooner bought by paying USD. Assume all transactions happen simultaneously, and the only restriction is that Richy should have nonnegative numbers of Crooners and USD at the end of all transactions. Formulate an LP that maximizes the total number of USD Richy has after all transactions. Graphically solve the LP, and comment on the solution.

d.V.'s $x_1 = \#$ USD, $x_2 = \#$ Crs (Crooners) at end

$$\max \quad z = x_1 - \frac{1}{5}x_2 \qquad (\#\text{ USD remaining})$$

$$x_1 - \frac{1}{5}x_2 \geq 0 \qquad (\text{nonneg }\#\text{ USD})$$

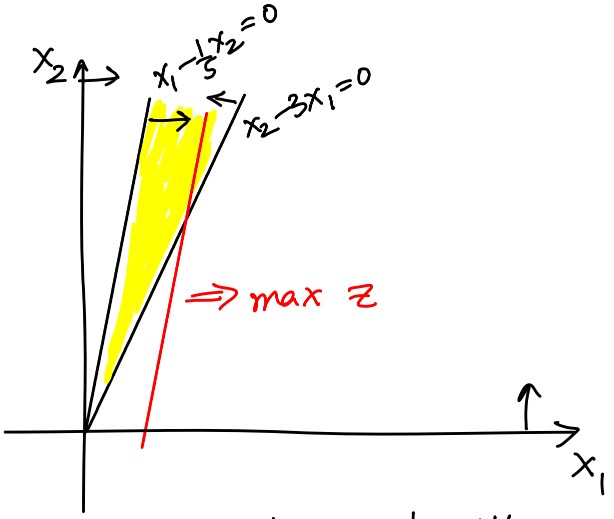
$x_2 \leq 5x_1$ ←

$$x_2 - 3x_1 \geq 0 \qquad (\text{nonneg }\#\text{ Crooners})$$

$x_2 \geq 3x_1$ →

$$x_1, x_2 \geq 0$$

Feasible region extends upward without limit, and we could slide the z-line to the right without limit. Case 4 — unbounded LP.



The exchange rates given here are unreasonable, and will never be seen in real life.

$1 \text{ USD} \xrightarrow{\times 5} 5 \text{ Crs} \xrightarrow{\times \frac{1}{3}} \frac{5}{3} \text{ USD}.$ ⤳ So Richy could become infinitely rich!

MATH 364: Lecture 6 (09/05/2024)

Today: LP formulations

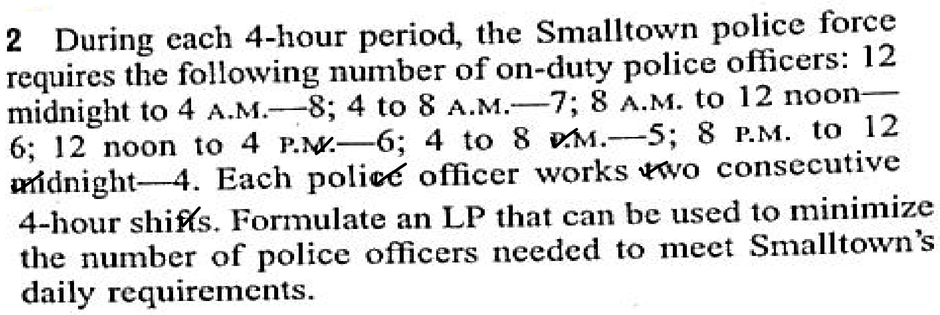
Linear Programming Formulation Problems

We introduce several LP formulation problems. They represent various scenarios from different application areas. The more such problems you are exposed to, the more comfortable you will get in tackling them.

Staffing Problems

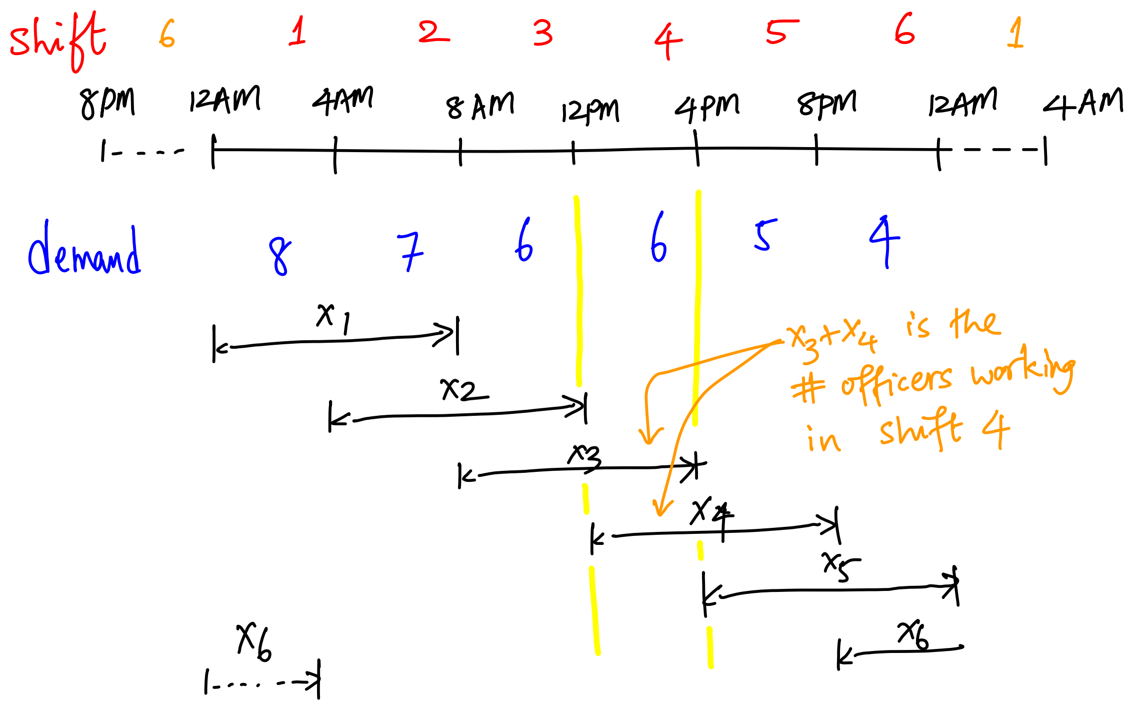
→ Winston-Venkataramanan: Intro to Math Programming

WV-IMP problem 2 Pg 75:



2 During each 4-hour period, the Smalltown police force requires the following number of on-duty police officers: 12 midnight to 4 A.M.—8; 4 to 8 A.M.—7; 8 A.M. to 12 noon—6; 12 noon to 4 P.M.—6; 4 to 8 P.M.—5; 8 P.M. to 12 midnight—4. Each police officer works two consecutive 4-hour shifts. Formulate an LP that can be used to minimize the number of police officers needed to meet Smalltown's daily requirements.

A sketch of the timeline (as shown below) is often helpful for such problems.



d.v.s

Let x_i = # officers starting duty in shift i, $i = 1, \ldots, 6$
(x_6 = # officers working shifts 6 and 1 of next day)

$$
\begin{array}{llcccccccll}
\min & z = & x_1 & + & x_2 + x_3 + x_4 + & x_5 & + & x_6 & & & \text{(total \# officers)} \\
\text{s.t.} & & x_1 & & & & + & x_6 & \geq & 8 & \text{(shift 1 req.)} \\
 & & x_1 & + & x_2 & & & & \geq & 7 & \text{(shift 2 req.)} \\
 & & & & x_2 + x_3 & & & & \geq & 6 & \text{(shift 3 req.)} \\
 & & & & x_3 + x_4 & & & & \geq & 6 & \text{(shift 4 req.)} \\
 & & & & x_4 + & x_5 & & & \geq & 5 & \text{(shift 5 req.)} \\
 & & & & & x_5 & + & x_6 & \geq & 4 & \text{(shift 6 req.)} \\
 & & & & & & & x_j & \geq & 0,\ j=1,\ldots,6 & \text{(non-neg)}
\end{array}
$$

subject to

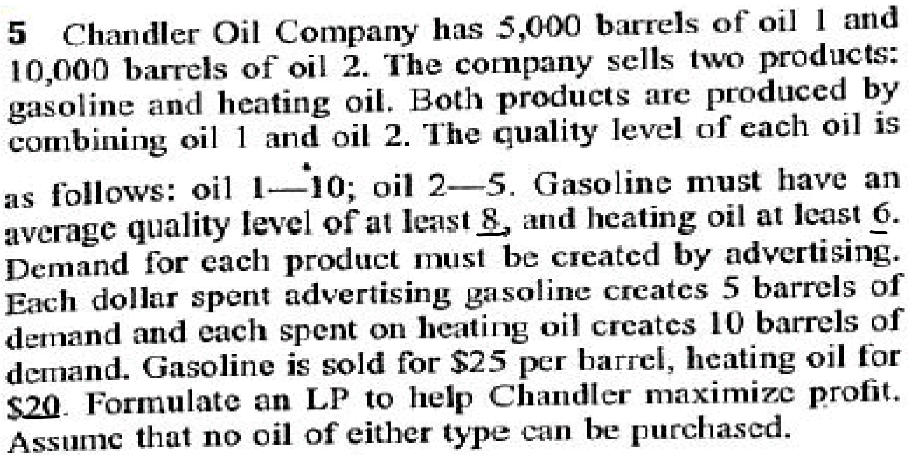
It is not necessary to line up the columns of each x_j when you write such formulations. But this practice does help with the readability!

A point about the divisibility assumption for the Smalltown police LP. Ideally we need to insist that all x_j's are integers, as they model # cops. But in this case, the optimal solution will have integer values. This happens because a special property is satisfied. We will revisit this topic later.

Blending Problems

→ a class of problems where several raw materials are blended together to form products.

WV-IMP Problem 5, pg 92:



5 Chandler Oil Company has 5,000 barrels of oil 1 and 10,000 barrels of oil 2. The company sells two products: gasoline and heating oil. Both products are produced by combining oil 1 and oil 2. The quality level of each oil is as follows: oil 1—10; oil 2—5. Gasoline must have an average quality level of at least 8, and heating oil at least 6. Demand for each product must be created by advertising. Each dollar spent advertising gasoline creates 5 barrels of demand and each spent on heating oil creates 10 barrels of demand. Gasoline is sold for $25 per barrel, heating oil for $20. Formulate an LP to help Chandler maximize profit. Assume that no oil of either type can be purchased.

Assume conservation of mass/volume, e.g., 10 ba oil 1 + 15 ba oil 2 ≡ 25 ba gas

Assumption about blending: the volume of product is equal to the volumes of crudes (or raw materials) mixed, i.e., there is no volume lost.

	quality	ad	price	oil 1	oil 2
gas	≥ 8	5 ba/$	$25/ba	qlty: 10	5
heating oil	≥ 6	10 ba/$	$20/ba	qty: 5,000	10,000

decisions

1. how much gas and h.oil to make?
2. how much to spend on ads for gas & h.oil?

   1a. # barrels of oil 1 & oil 2 used for making gas?
   1b. # barrels of oil 1 & oil 2 used for making h.oil?

d.v.'s

x_{ij} = # barrels of oil i used to make product j, $i=1,2$
$j=g,h$
gas h.oil

$x_{1g}, x_{1h}, x_{2g}, x_{2h}$

y_j = $ spent on ads for product j, $j=g,h$ (y_g, y_h)

Objective function (maximize profit)

$$\max \ z = 25\underbrace{(x_{1g}+x_{2g})}_{\text{total \# ba. of gas}} + 20\underbrace{(x_{1h}+x_{2h})}_{\text{total vol. of h.oil}} - \underbrace{(y_g+y_h)}_{\text{cost for ads}} \quad \text{(net profit)}$$

We could use extra variables to model the total amount of gas and total amount of heating oil. But we still need the split variables x_{ig}, x_{ih}, $i=1,2$.

Constraints

Limit on availability of oils 1 and 2:

$$x_{1g} + x_{1h} \leq 5000 \quad \text{(oil 1 avail.)}$$

$$x_{2g} + x_{2h} \leq 10{,}000 \quad \text{(oil 2 avail.)}$$

Meet demand generated by ads:

$$x_{1g} + x_{2g} \geq 5\underbrace{y_g}_{} \quad \text{(demand for gas)}$$

barrels of demand for gas generated by spending $\$y_g$ on ads

$$x_{1h} + x_{2h} \geq 10 y_h \quad \text{(demand for h.oil)}$$

Average quality:

$$\underbrace{\frac{10 \cdot x_{1g} + 5 \cdot x_{2g}}{x_{1g} + x_{2g}}}_{} \geq 8 \quad \text{(quality of gas)}$$

average quality of gasoline - the average is taken over the volume mixed, as a weighted average

$$\frac{10 \cdot X_{1h} + 5 \cdot X_{2h}}{X_{1h} + X_{2h}} \geq 6 \quad \text{(quality of h.oil)}$$

You could leave these constraints as is — no need to simplify.

Note that the quality constraints are indeed linear: cross multiply!

$$\frac{10 \cdot X_{1g} + 5 \cdot X_{2g}}{X_{1g} + X_{2g}} \geq 8 \Rightarrow \quad 10X_{1g} + 5X_{2g} \geq 8(X_{1g} + X_{2g}) \Rightarrow$$

$$2X_{1g} - 3X_{2g} \geq 0.$$

But you need not necessarily do this step of simplification - as long as you are sure of the linearity. In fact, the software AMPL will do the simplification for you!

Sign restrictions: $x_{ij}, y_j \geq 0$ for all i, j (non-neg)

or all vars ≥ 0 (non-neg).

Here is the entire LP:

$$\max \ Z = 25(X_{1g} + X_{2g}) + 20(X_{1h} + X_{2h}) - (y_g + y_h) \quad \text{(net profit)}$$

s.t.

$$X_{1g} + X_{1h} \leq 5000 \quad \text{(oil 1 avail.)}$$

$$X_{2g} + X_{2h} \leq 10{,}000 \quad \text{(oil 2 avail.)}$$

$$X_{1g} + X_{2g} \geq 5y_g \quad \text{(demand for gas)}$$

$$X_{1h} + X_{2h} \geq 10y_h \quad \text{(demand for h.oil)}$$

$$(10 \cdot X_{1g} + 5 \cdot X_{2g})/(X_{1g} + X_{2g}) \geq 8 \quad \text{(quality of gas)}$$

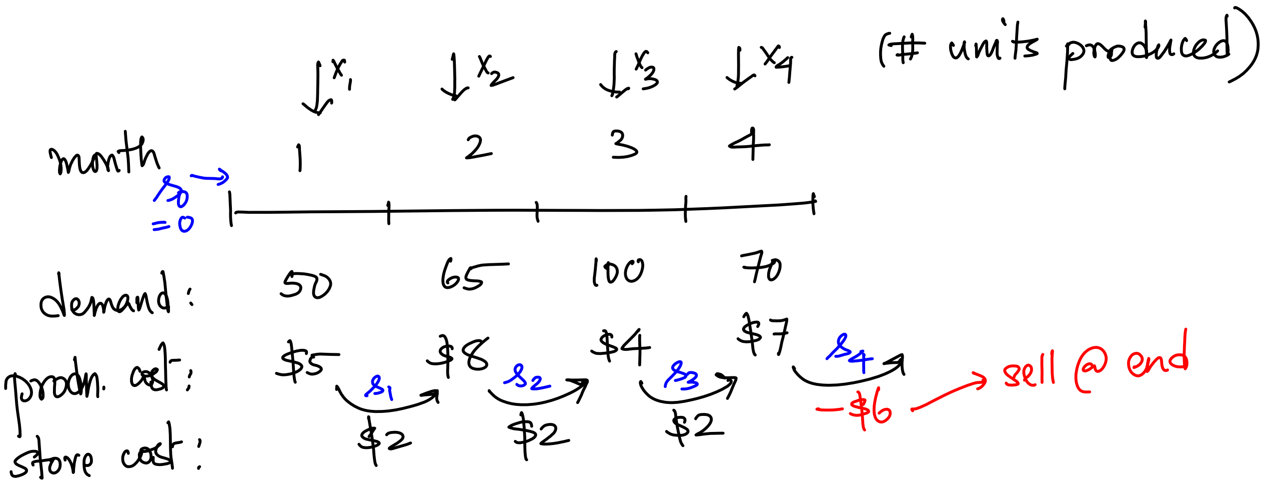
$$(10 \cdot X_{1h} + 5 \cdot X_{2h})/(X_{1h} + X_{2h}) \geq 6 \quad \text{(quality of h.oil)}$$

$$\text{all vars} \geq 0 \quad \text{(non-neg)}$$

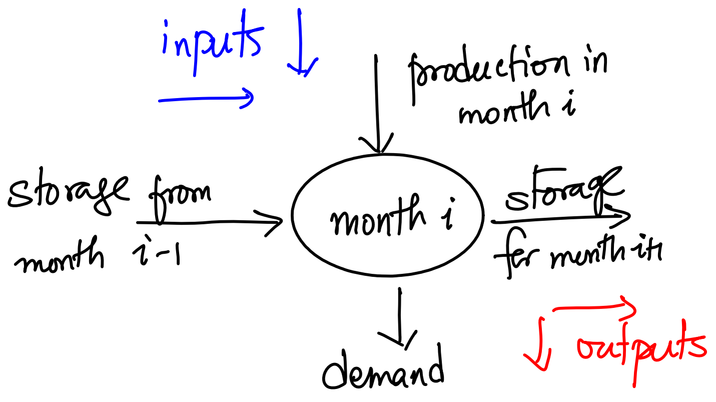
Inventory Planning

WV-IMP Pg 104, Problem 1:

1 A customer requires during the next four months, respectively, 50, 65, 100, and 70 units of a commodity (no backlogging is allowed). Production costs are \$5, \$8, \$4, and \$7 per unit during these months. The storage cost from one month to the next is \$2 per unit (assessed on ending inventory). It is estimated that each unit on hand at the end of month 4 could be sold for \$6. Formulate an LP that will minimize the net cost incurred in meeting the demands of the next four months.



For each month, we have "inflow = outflow" restriction, i.e., a flow-balance constraint.



We could produce some units in month i, and/or carry some over from month (i-1). All these items are used to satisfy demand in month i, and whatever is left is carried over to month i+1.

(we'll finish the formulation in the next lecture...)

MATH 364: Lecture 7 (09/10/2024)

Today: * one more formulation problem
* AMPL

We first finish the inventory planning LP...

d.v.'s

let x_i = # units produced in month i, $i = 1, \dots 4$

s_i = # units stored from month i to $i+1$, $i = 0, \dots, 4$

s_0 = starting inventory $(=0)$.

→ The problem did not mention anything about units available at start of month 1. We capture this quantity in s_0 — and can write all flow-balance constraints in a unified manner using s_0 (and s_i, x_i for $i = 1-4$).

Objective function

$$\min \quad z = \underbrace{5x_1 + 8x_2 + 4x_3 + 7x_4}_{\text{prodn cost}} + \underbrace{2s_1 + 2s_2 + 2s_3}_{\text{storage cost}} - \underset{\text{revenue at end}}{6s_4} \quad \text{(total cost)}$$

Constraints

$s_0 + x_1 = 50 + s_1$ (inventory balance month 1)

$s_1 + x_2 = 65 + s_2$ (inv. balance month 2)

$s_2 + x_3 = 100 + s_3$ (inv. balance month 3)

$s_3 + x_4 = 70 + s_4$ (inv. Balance month 4)

$s_0 = 0$ (no starting inventory)

$s_i, x_i \geqslant 0$, for all i (non-neg).

If we let d_i be the demand in month i (this is data given to us), we could write the balance constraints for all months in one go:

$$\underbrace{s_{i-1} + x_i}_{\text{inflow}} = \underbrace{d_i + s_i}_{\text{outflow}}, \quad i=1-4 \qquad (\text{inv. balance month } i)$$

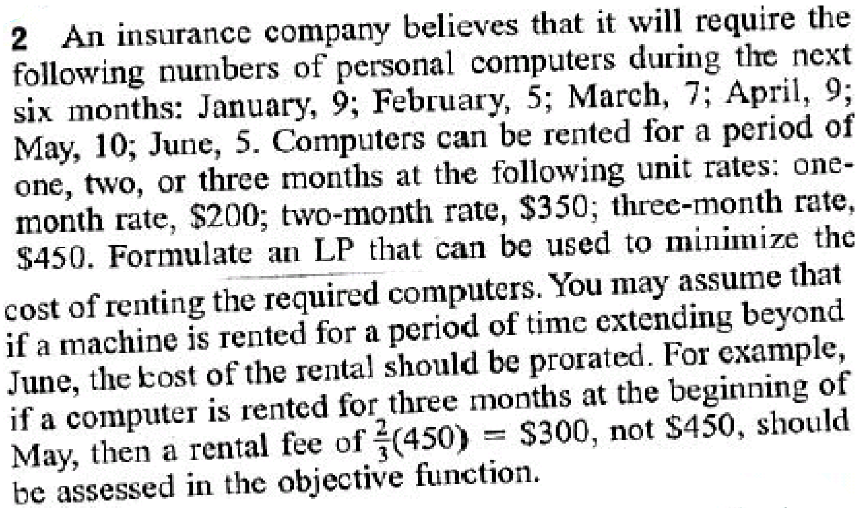
Here, $d_1 = 50$, $d_2 = 65$, etc.

Also note that we do not have to write additional constraints that ensure all demand is met. We would write

$$\underbrace{s_{i-1} + x_i - s_i}_{\text{net inflow}} \geq d_i \qquad (\text{meet demand month } i)$$

But this constraint is forced (at equality) by the balance constraints. And we do need the balance constraints, so we could skip the demand constraints once the flow balance constraints are written.

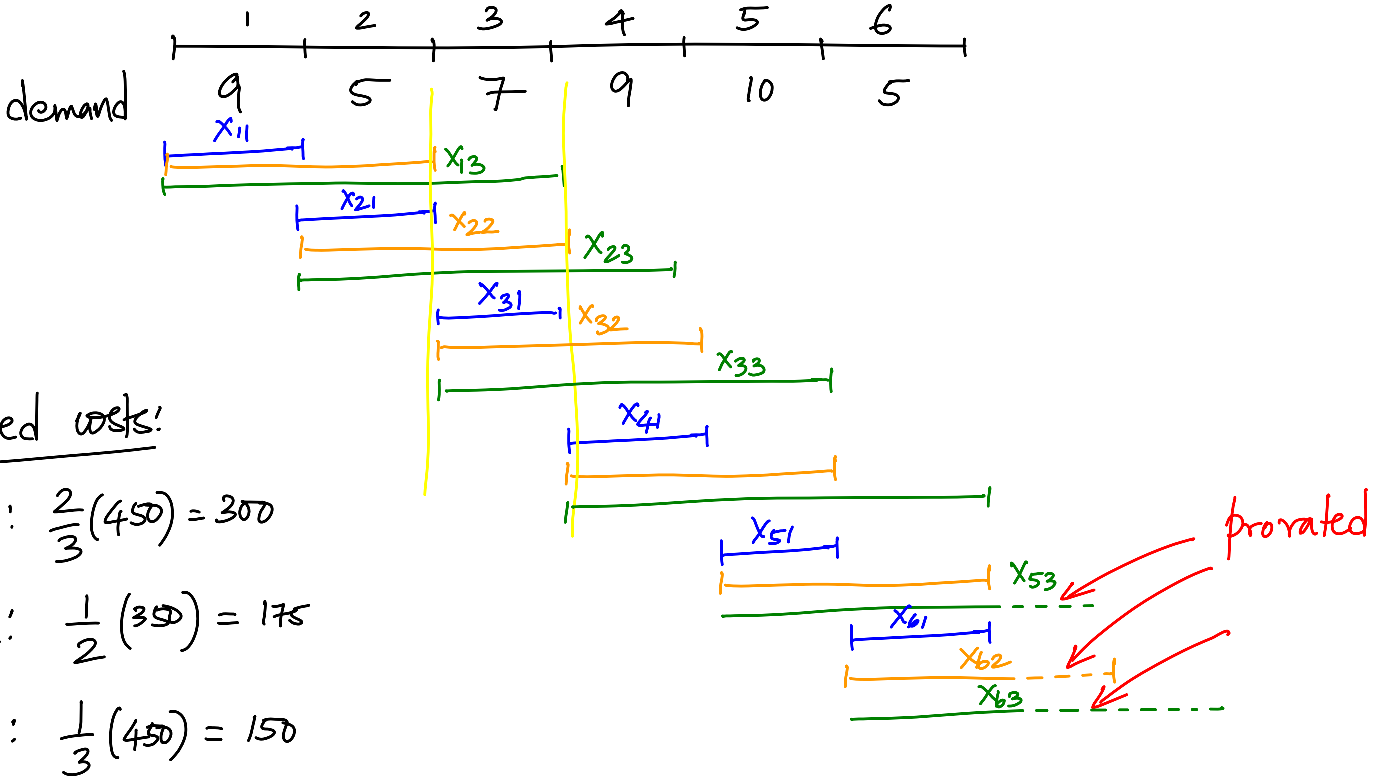
We consider a final formulation LP that is similar in flavor to the Small Town Police scheduling LP



2 An insurance company believes that it will require the following numbers of personal computers during the next six months: January, 9; February, 5; March, 7; April, 9; May, 10; June, 5. Computers can be rented for a period of one, two, or three months at the following unit rates: one-month rate, \$200; two-month rate, \$350; three-month rate, \$450. Formulate an LP that can be used to minimize the cost of renting the required computers. You may assume that if a machine is rented for a period of time extending beyond June, the cost of the rental should be prorated. For example, if a computer is rented for three months at the beginning of May, then a rental fee of $\frac{2}{3}(450) = \$300$, not \$450, should be assessed in the objective function.

d.v.'s

Let x_{ij} = # computers rented starting in month i on a j-month lease,
$i=1,\dots,6$, $j=1,2,3$.



Prorated costs:

x_{53}: $\frac{2}{3}(450) = 300$

x_{62}: $\frac{1}{2}(350) = 175$

x_{63}: $\frac{1}{3}(450) = 150$

Allowed to prorate only @ end!

Hint on Problem 1 (HW3):

x_{13} = # officers working shifts 1 and 3; similarly,

$x_{14}, x_{15}, x_{16}, \quad x_{24}, \ldots, x_{46}$.

In general: $x_{i,i+1}$ and $x_{i,j}$ for $j > i+1$

= # officers working shifts $i, i+1$ or i and j.

Here is the computer leasing problem:

$$\min\ z = 200\Big(\overset{\substack{\text{1-month leases}\\ \downarrow}}{\sum_{i=1}^{6}} x_{i1}\Big) + 350\Big(\overset{\substack{\text{2-month leases}\\ \downarrow}}{\sum_{i=1}^{5}} x_{i2}\Big) + 450\Big(\overset{\substack{\text{3-month leases}\\ \downarrow}}{\sum_{i=1}^{4}} x_{i3}\Big) +$$

$$\frac{2}{3}(450)\,x_{53} + \frac{1}{2}(350)\,x_{62} + \frac{1}{3}(450)\,x_{63} \qquad \text{(total cost)}$$

$$\begin{array}{lrcrl}
\text{s.t.} & x_{11} + x_{12} + x_{13} & \geq & 9 & \text{(Jan demand)} \\
 & x_{12} + x_{13} + x_{21} + x_{22} + x_{23} & \geq & 5 & \text{(Feb demand} \\
 & x_{13} + x_{22} + x_{23} + x_{31} + x_{32} + x_{33} & \geq & 7 & \text{(Mar demand)} \\
 & x_{23} + x_{32} + x_{33} + x_{41} + x_{42} + x_{43} & \geq & 9 & \text{(Apr demand)} \\
 & x_{33} + x_{42} + x_{43} + x_{51} + x_{52} + x_{53} & \geq & 10 & \text{(May demand)} \\
 & x_{43} + x_{52} + x_{53} + x_{61} + x_{62} + x_{63} & \geq & 5 & \text{(Jun demand)} \\
 & \text{all } x_{ij} & \geq & 0 & \text{(non-neg)}
\end{array}$$

See the AMPL handout, AMPL session, and the lecture video ...

MATH 364: Lecture 9 (09/17/2024)

Today: * LP in standard form
* basic solutions, bfs

We will introduce the **simplex algorithm** to solve LPs with multiple variables using EROs. To apply this method, we first need to convert the LP into a standard $A\bar{x}=\bar{b}$ form — note that the input LP could have $\geq, \leq$, or $=$ constraints to start with.

Def An LP is in **standard form** if

1. all constraints are of the "$=$" form (equations); and
2. all variables are nonnegative (≥ 0).

The objective function could be min or max.

So, no ≤ 0 or unrestricted in sign variables in standard form

Let's convert the following LP to standard form:

$$\begin{array}{lrl} \min & z = & 3x_1 + x_2 \\ \text{s.t.} & x_1 & \geq 3 \\ & \boxed{x_1 + x_2 \leq 4} & \\ & 2x_1 - x_2 & = 3 \\ & x_1, x_2 & \geq 0 \end{array}$$

Let's consider this constraint first.

We convert $x_1+x_2 \leq 4$ to an equation by adding a **slack variable** s to the left-hand side, and adding non-negativity for s.

$$\begin{aligned} x_1 + x_2 + s &= 4 \\ s &\geq 0 \end{aligned}$$

Note that $s \geq 0$ is required here. If $s=-1$, for instance, $x_1+x_2=5$, which violates the original constraint.

Recall: Farmer Jones LP:

$$\begin{aligned} x_1 + x_2 &\leq 7 \quad \text{(land available)} \\ 4x_1 + 10x_2 &\leq 40 \quad \text{(labor hrs)} \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + \overset{\text{\# acres unused}}{s_1} \qquad &= 7 \\ 4x_1 + 10x_2 \qquad + \underset{\text{\# labor hrs unused}}{s_2} &= 40 \\ s_1, s_2 &\geq 0 \end{aligned}$$

$s_i \rightarrow$ slack variable for the i^{th} constraint

Now, for $x_1 \geq 3$, we can write

$x_1 - e = 3$ and add $e \geq 0$.

Here e is the **excess variable** (or surplus variable).

$$\begin{array}{ll} \min & z = 3x_1 + x_2 \\ \text{s.t.} & x_1 \geq 3 \\ & x_1 + x_2 \leq 4 \\ & 2x_1 - x_2 = 3 \\ & x_1, x_2 \geq 0 \end{array} \longrightarrow \underbrace{\begin{array}{ll} \min & z = 3x_1 + x_2 \\ \text{s.t.} & x_1 - e = 3 \\ & x_1 + x_2 + s = 4 \\ & 2x_1 - x_2 = 3 \\ & x_1, x_2, s, e \geq 0 \end{array}}_{\text{LP in standard form}}$$

Slack/excess variables do not show up in the objective function.

↳ or, they have coefficient zero in the objective function.

Another Example

$$
\begin{array}{ll}
\max & z = 20x_1 + 15x_2 \\
\text{s.t.} & \begin{array}{rcll}
x_1 & \le & 100 & s_1 \\
x_2 & \le & 200 & s_2 \\
50x_1 + 35x_2 & \le & 5000 & s_3 \\
25x_1 + 15x_2 & \ge & 2000 & e_4 \\
x_1, x_2 & \ge & 0 &
\end{array}
\end{array}
$$

It is just convenient notation to use s_i for slack variable of i^{th} constraint, and e_j for the excess variable of j^{th} constraint.

↳ But one could instead use x_3, x_4, x_5, x_6 as the slack/excess variables here!

$$
\begin{array}{ll}
\max & z = 20x_1 + 15x_2 \\
\text{s.t.} & \begin{array}{rcl}
x_1 + s_1 & = & 100 \\
x_2 + s_2 & = & 200 \\
50x_1 + 35x_2 + s_3 & = & 5000 \\
25x_1 + 15x_2 - e_4 & = & 2000 \\
x_1, x_2, s_1, s_2, s_3, e_4 & \ge & 0
\end{array}
\end{array}
$$

If input LP has $A\bar{x} \begin{pmatrix} \le \\ \ge \\ \vdots \end{pmatrix} \bar{b}$, then the standard form LP will be $[A\, I']\bar{x}' = \bar{b}$ where I' is "almost identity" matrix, and $\bar{x}' = \begin{bmatrix} \bar{x} \\ \bar{s} \end{bmatrix}$

original vars

slack/excess vars.

If all constraints are $\le$, then we get $[A\; I]$, where I is the $m \times m$ identity matrix (assuming there are m constraints).

We will talk about the second condition (of requiring all variables to be ≥ 0) later on. For now, assume all variables are ≥ 0 to start with.

Once in the standard form, notice that the constraints all form a system $A\bar{x}=\bar{b}$. If the system has a unique solution, there is nothing more to do — that solution is the optimal solution. If $A\bar{x}=\bar{b}$ is inconsistent, then the LP is infeasible. The interesting case happens when $A\bar{x}=\bar{b}$ has free variables, and then we will involve the objective function to choose a best solution from among the infinitely many solutions possible.

An LP in standard form is
$$\begin{aligned} \max\ & \bar{c}^T\bar{x} \\ \text{s.t. } & A\bar{x}=\bar{b} \\ & \bar{x}\geq\bar{0} \end{aligned}$$

<u>Recall</u>: GJ for solving $A\bar{x}=\bar{b}$, $A\in\mathbb{R}^{m\times n}$, $\operatorname{rank}(A)=m\leq n$.

$$[A\,|\,\bar{b}] \longrightarrow [B\ N\,|\,\bar{b}] \xrightarrow{EROs} [I_m\ \tilde{N}\,|\,\tilde{\bar{b}}]$$

With $\bar{x}_B$ as the m basic variables, and $\bar{x}_N$ as the $(n-m)$ non-basic (or free) variables, we get with $\bar{x}=\begin{bmatrix}\bar{x}_B\\ \bar{x}_N\end{bmatrix}$

$$\bar{x}_B+\tilde{N}\bar{x}_N=\tilde{\bar{b}}$$

$$\Rightarrow\quad \bar{x}_B=\tilde{\bar{b}}-\tilde{N}\bar{x}_N$$

Choosing $\bar{x}_N=\bar{\alpha}$ (vector of parameters), we get $\bar{x}_B=\tilde{\bar{b}}-\tilde{N}\bar{\alpha}$.

The solution $\bar{x}$ obtained by setting $\bar{x}_N=\bar{\alpha}=\bar{0}$ is called a **basic solution** of $A\bar{x}=\bar{b}$.

$\bar{x}_N=\bar{0} \Rightarrow \bar{x}_B=\tilde{\bar{b}}$, so $\bar{x}=\begin{bmatrix}\tilde{\bar{b}}\\ \bar{0}\end{bmatrix}$ is a basic solution.

How to find (a) basic solution(s)?

1. Choose m basic variables (BV), which correspond to m LI columns of A.

when $n > m$, there could be many subsets of m vars that are basic.

2. Set the remaining $(n-m)$ non-basic vars (NBV) to 0.
3. Solve for the m basic variables.

Example

$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad m=2, \quad n=3$$

1. $BV = \{x_1, x_2\}$, $NBV = \{x_3\}$. Set $x_3 = 0$, solve for x_1, x_2:

$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 &= -1 \end{aligned} \qquad x_1 = 2, x_2 = 1$$

So, Basic solution is $\bar{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

2. $BV = \{x_1, x_3\}$, $NBV = \{x_2\}$.

Basic solution is $\bar{x} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$

if the input system were that of an LP, this solution violates feasibility, as $x_3 \ngeq 0$.

3. $BV = \{x_2, x_3\}$, $NBV = \{x_1\}$. $x_1 = 0$

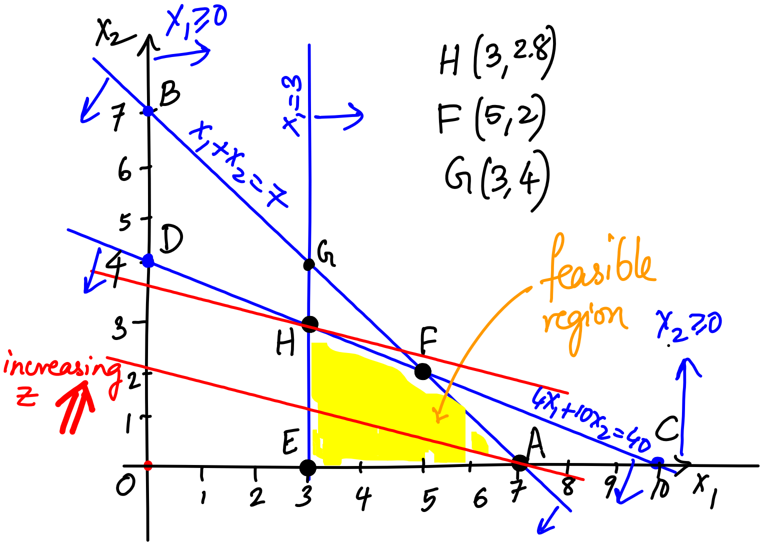
Basic solution is $\bar{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$.

Def For an LP in standard form, a basic solution in which all variables are nonnegative is a **basic feasible solution** (bfs).

Why study bfs's?

Recall: feasible region of an LP is a convex set.

Result If an LP in standard form has an optimal solution, then a corner point is guaranteed to be optimal.



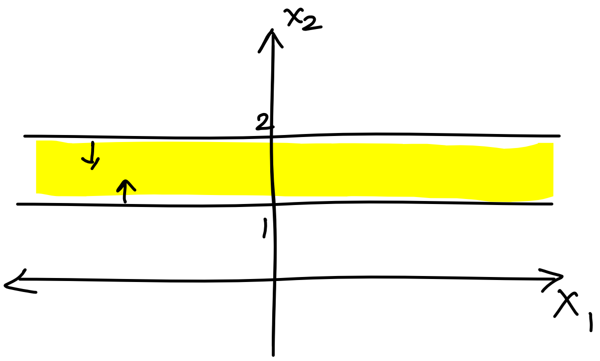
Q. Does every feasible LP have a corner point? No!

Consider max x_2

s.t. $1 \leq x_2 \leq 2$

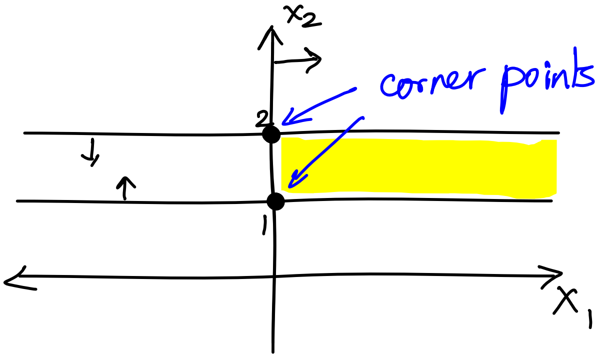
x_1 urs

unrestricted in sign
could be ≥ 0 or ≤ 0



There are no corner points here. The LP is indeed not unbounded. Any (x_1, x_2) with $x_2 = 2$ is an optimal solution (Case 2).

If we add $x_1 \geq 0$, we get an LP in standard form, and we get two corner points!



Result Every LP in standard form has corner point(s).

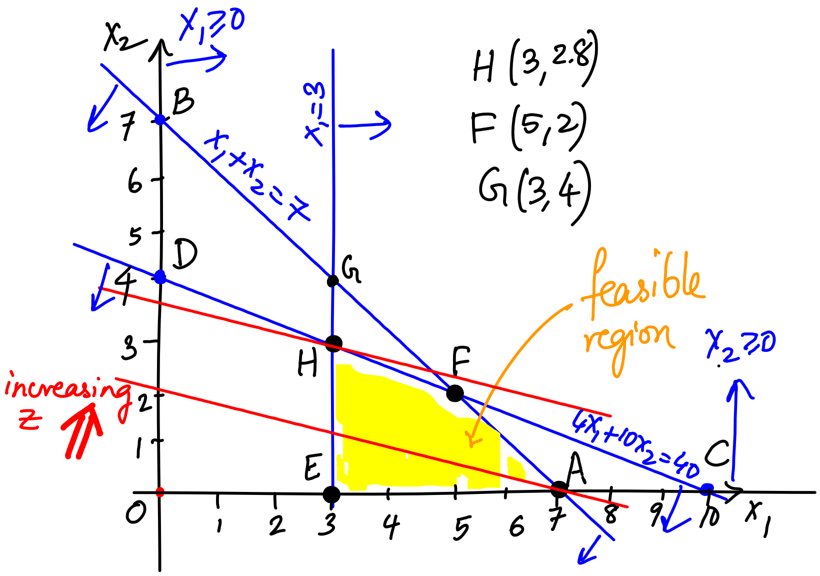
Result A point in the feasible region of an LP in standard form is a corner point <u>if and only if</u> it corresponds to a bfs.

So, **corner point ≡ bfs.**

We demonstrate this correspondence for the Farmer Jones LP:

Farmer Jones LP

$$
\begin{aligned}
\max\ z &= 30x_1 + 100x_2 \\
\text{s.t.}\quad x_1 + x_2 &\le 7 \qquad s_1 \\
4x_1 + 10x_2 &\le 40 \qquad s_2 \\
10x_1 &\ge 30 \qquad e_3 \\
x_1, x_2 &\ge 0
\end{aligned}
$$



Standard form

$$
\begin{aligned}
\max\ z &= 30x_1 + 100x_2 \\
\text{s.t.}\quad x_1 + x_2 + s_1 &= 7 \\
4x_1 + 10x_2 + s_2 &= 40 \\
10x_1 - e_3 &= 30 \\
x_1, x_2, s_1, s_2, e_3 &\ge 0
\end{aligned}
$$

$m = 3,\ n = 5$

$\text{rank} = m = 3$

$$\begin{aligned} x_1 + x_2 + s_1 \quad\quad &= 7 \\ 4x_1 + 10x_2 \quad + s_2 \quad &= 40 \\ 10x_1 \quad\quad - e_3 &= 30 \end{aligned}$$

1. $BV = \{x_1, x_2, s_1\}$, $NBV = \{s_2, e_3\}$

Setting $s_2 = e_3 = 0$, and solving we get $x_1 = 3$, $x_2 = 2.8$, $s_1 = 1.2$.

$$\bar{x} = \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 2.8 \end{matrix}} \xrightarrow{} H(3, 2.8) \\ 1.2 \\ 0 \\ 0 \end{bmatrix}$$

is a bfs, and corresponds to the vertex $H(3, 2.8)$

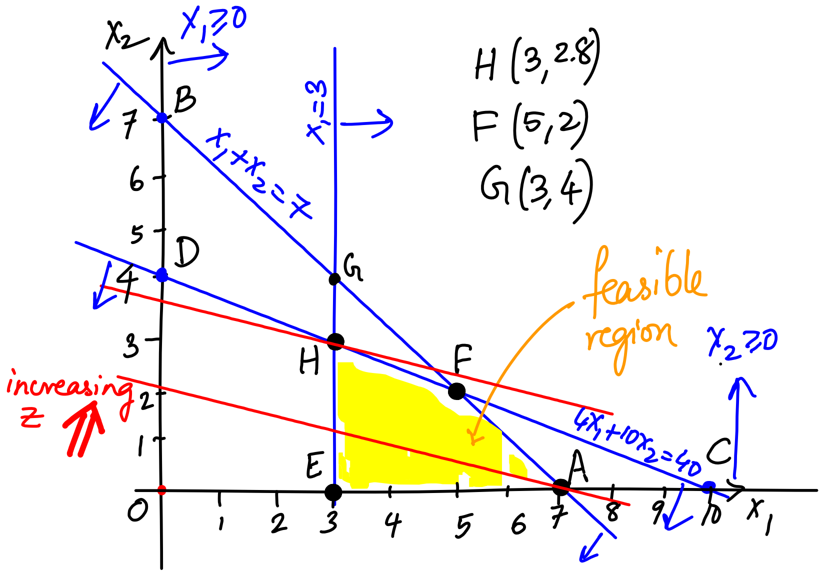
MATH 364: Lecture 10 (09/19/2024)

Today:
- * correspondence between bfs's & corner points
- * simplex method

Farmer Jones LP

$$\max \ z = 30x_1 + 100x_2$$

$$\text{s.t.} \quad \begin{aligned} x_1 + x_2 + s_1 &= 7 \\ 4x_1 + 10x_2 + s_2 &= 40 \\ 10x_1 - e_3 &= 30 \\ x_1, x_2, s_1, s_2, e_3 &\geq 0 \end{aligned}$$



We saw $BV = \{x_1, x_2, s_1\}$, $NBV = \{s_2, e_3\}$ gives the bfs $\equiv H(3, 2.8)$.

$$\bar{x} = \begin{array}{c} x_1 \\ x_2 \\ s_1 \\ s_2 \\ e_3 \end{array} \begin{bmatrix} 3 \\ 2.8 \\ 1.2 \\ 0 \\ 0 \end{bmatrix}.$$

For $BV = \{x_2, s_2, e_3\}$, $NBV = \{x_1, s_1\}$ gives the basic solution

$$\bar{x} = \begin{array}{c} x_1 \\ x_2 \\ s_1 \\ s_2 \\ e_3 \end{array} \begin{bmatrix} 0 \\ 7 \\ 0 \\ -30 \\ -30 \end{bmatrix}$$

→ corresponds to B(0, 7), which is not feasible.

$$\left\{ \begin{aligned} x_2 &= 7 \\ 10x_2 + s_2 &= 40 \\ -e_3 &= 30 \end{aligned} \right\} \Longrightarrow x_2 = 7, \ e_3 = -30, \ s_2 = -30.$$

We could identify the bfs corresponding to each corner point directly from the picture!

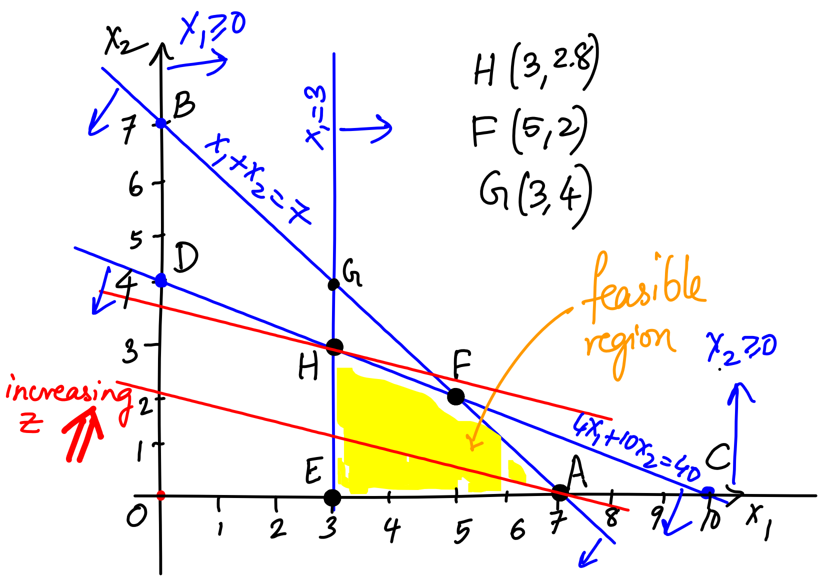
Let's consider F(5,2).

$x_1=5$, $x_2=2$ are in BV.

At F, the (land_avai) and (labor_hrs) constraints are binding (i.e., satisfied as equalities). Hence $s_1=0$ and $s_2=0$. But the (min_corn) constraint is non-binding at F, hence $e_3>0$. Hence $BV=\{x_1,x_2,e_3\}$, $NBV=\{s_1,s_2\}$, and the bfs is

$$X = \begin{array}{c|c} x_1 & 5 \\ x_2 & 2 \\ s_1 & 0 \\ s_2 & 0 \\ e_3 & 20 \end{array} \;.$$

$\rightarrow 10(5)-30$



We can present the correspondence between corner points and bfs's in a table as shown below.

Correspondence between bfs's and corner points

Corner point	BV	NBV	BFS $[x_1\ x_2\ s_1\ s_2\ e_3]$
A(7,0)	x_1, e_3, s_2	x_2, s_1	[7 0 0 12 40]
E(3,0)	x_1, s_1, s_2	x_2, e_3	[3 0 4 28 0]
H(3, 2.8)	x_1, x_2, s_1	s_2, e_3	[3 2.8 1.2 0 0]
F(5, 2)	x_1, x_2, e_3	s_1, s_2	[5 2 0 0 20]

Let's summarize a bit. We have seen the following results.

- * If the LP in standard form has an optimal solution, there must be a corner point that is optimal.
- * Corner points $\Longleftrightarrow$ bfs

Hence we get the following result.

Theorem If an LP in standard form has an optimal solution, then it has an optimal bfs.

The simplex method explores the corner points, or bfss. The idea is to start at one bfs, and move to a neighboring bfs (or corner point) at which the objective function is better. This procedure is equivalent to sliding the z-line in 2D.

While the idea of a neighboring corner point is straightforward to imagine in 2D, we switch to an algebraic view in higher dimensions using the correspondence given above.

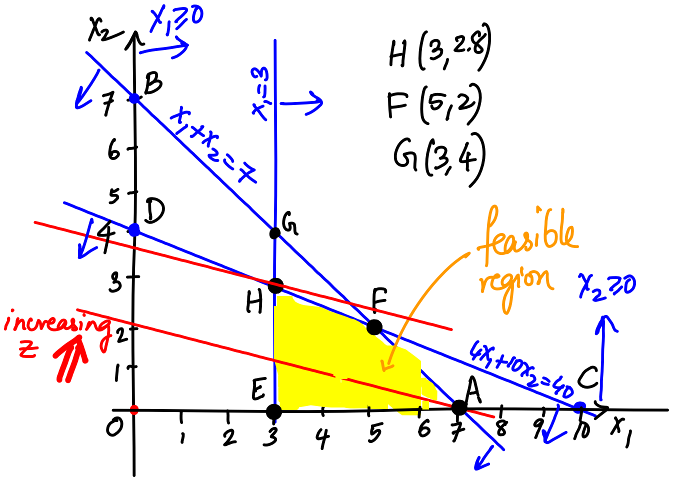
Simplex Method

- * Start at a bfs $\equiv$ corner point.
- * If not optimal, move to a "nearby" (adjacent) bfs so that the z-value improves
- * If no such "better" adjacent bfs exists, the current corner point $\equiv$ bfs is optimal.

Adjacent bfs

Def for an LP in standard form with n variables and m constraints ($m \leq n$, $\text{rank}(A) = m$), two bfs's are said to be **adjacent** if they have $(m-1)$ common basic variables.

Corner point	BV	NBV	BFS $[x_1\ x_2\ s_1\ s_2\ e_3]$
A (7,0)	x_1, e_3, s_2	x_2, s_1	$[7\ 0\ 0\ 12\ 40]$
E (3,0)	x_1, s_1, s_2	x_2, e_3	$[3\ 0\ 4\ 28\ 0]$
H(3, 2.8)	x_1, x_2, s_1	s_2, e_3	$[3\ 2.8\ 1.2\ 0\ 0]$
F (5,2)	x_1, x_2, e_3	s_1, s_2	$[5\ 2\ 0\ 0\ 20]$



For instance, we could start at A (7,0), where $z = 210$, and move to the adjacent bfs F(5,2), where $z = 350$. Notice that the other adjacent bfs to A is E(3,0). But $z = 90$ at E, and hence we do not move to E.

Repeating the same procedure, we move from F to the adjacent bfs H(3,2.8), where $z = 370$. From H, both adjacent corner points (F and E) have smaller z-values, and hence we can conclude that H is an optimal solution.

We could, alternatively start at E and move to H directly.

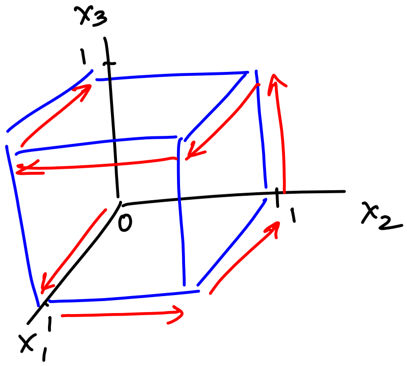
How many bfs's are there?

Could solve LP by evaluating z at every bfs (corner point).
What is the max # bfs's an LP can have?

For each bfs we need to choose m basic variables out of n variables.
Hence there are $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ choices, which could be quite large!

But not all of these choices might lead to a bfs

On the one hand, there are artificially constructed LP instances for which every version of the simplex algorithm would have to inspect all of $\binom{n}{m}$ bfss (all of them being feasible, of course).



2^n vertices in general!

But on the other hand most LPs arising out of applications tend not to exhibit such structure, and the simplex method is usually very fast in solving most LPs.

Simplex Method

Simplex Algorithm for maximization LPs

Step 1 Convert LP to standard form.

Step 2 Obtain a bfs from the standard form.

Step 3 Find if current bfs is optimal.
If YES, STOP.

Step 4 If current bfs is not optimal, find which non-basic variable should become basic, and which basic variable should become non-basic in order to move to an adjacent bfs with a higher objective function value.
↳ we are solving a max LP.

Step 5 Use EROs to obtain the adjacent bfs.
Return to Step 3.

We specify more details for each step as we illustrate the simplex algorithm on an example. We start with an LP where all constraints are '$\leq$'. Step 2 becomes easy in this case. We will discuss how to deal with '$\geq$' and '$=$' constraints later on. We will assume also that all variables are non-negative for now.

Solve the following LP using the simplex method

$$
\begin{array}{ll}
\max & z = 2x_1 + 3x_2 \\
\text{s.t.} & x_1 + 2x_2 \le 6 \quad s_1 \geq 0 \\
 & 2x_1 + x_2 \le 8 \quad s_2 \geq 0 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Step 1

$$
\begin{array}{ll}
\max & z = 2x_1 + 3x_2 \\
\text{s.t.} & x_1 + 2x_2 + s_1 \qquad = 6 \\
 & 2x_1 + x_2 \qquad + s_2 = 8 \\
 & x_1, x_2, s_1, s_2 \geq 0
\end{array}
$$

Step 2 We first write the LP in a more organized manner:

$$
\left.
\begin{array}{lcccccl}
0. & z & -2x_1 & -3x_2 & & & = 0 \\
1. & & x_1 & +2x_2 & +s_1 & & = 6 \\
2 & & 2x_1 & +x_2 & & +s_2 & = 8
\end{array}
\right\} \text{canonical form}
$$

Def An LP is written in **canonical form** if each row including Row-0 has a variable (including z) with coefficient 1 in that row and zero in every other row.

Here, $\{z, s_1, s_2\}$ is the set of canonical variables.

We can choose z and the remaining m canonical variables in the starting bfs. If the rhs (b_i for the i^{th} constraint) values are all ≥ 0, we can read off the bfs from the canonical form.

Here, we set $x_1 = x_2 = 0$, and get $s_1 = 6$, $s_2 = 8$, and $z = 0$.

MATH 364: Lecture 11 (09/24/2024)

Today: * simplex for max LP
* tableau simplex

Simplex Algorithm for maximization LPs

Step 1 Convert LP to standard form.

Step 2 Obtain a bfs from the standard form.

Step 3 Find if current bfs is optimal.
If YES, STOP.

Step 4 If current bfs is not optimal, find which non-basic variable should become basic, and which basic variable should become non-basic in order to move to an adjacent bfs with a higher objective function value.

Step 5 Use EROs to obtain the adjacent bfs.
Return to Step 3.

Recall the steps of the simplex method for max-LP

We will continue with the example from Lecture 10:

$$\begin{aligned} \max\ & z = 2x_1+3x_2 \\ \text{s.t.}\ & x_1+2x_2 \le 6 \quad s_1 \ge 0 \\ & 2x_1+x_2 \le 8 \quad s_2 \ge 0 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

Step 1

$$\begin{aligned} \max\ & z = 2x_1+3x_2 \\ \text{s.t.}\ & x_1+2x_2+s_1 = 6 \\ & 2x_1+x_2+s_2 = 8 \\ & x_1,\ x_2,\ s_1,\ s_2 \ge 0 \end{aligned}$$

Step 2

$$\begin{array}{lrcl} 0. & z-2x_1-3x_2 & & = 0 \\ 1. & x_1+2x_2+s_1 & & = 6 \\ 2. & 2x_1+x_2 & +s_2 & = 8 \end{array}$$

Can read off the bfs from the LP in canonical form: Here, $s_1=6$, $s_2=8$ is the bfs, giving $z=0$.

$BV=\{z, s_1, s_2\}$, $NBV=\{x_1, x_2\}$.

Step 3 Check if current bfs is optimal.

bfs is optimal if we cannot improve the z-value by increasing the value of any non-basic variable (from 0).

Here, $z = 2x_1 + 3x_2 = 0$ now (right now $x_1 = x_2 = 0$).

If $x_1 = 1$, z becomes 2

If $x_2 = 1$, z becomes 3

} So, current bfs is not optimal.

We will see in Step 4 which of these two vars we will increase, and by how much.

Step 4 Want to move to an adjacent bfs such that the z-value increases.

We want to consider increasing one non-basic variable from 0 to a nonzero value, as we want to move to an adjacent bfs, which shares all but one basic variable with the current bfs.

We could increase either x_1 or x_2 to improve z. By default, we pick the non-basic variable that has the largest rate of increase – here, it's x_2. Hence x_2 is the **entering variable.**

x_2 enters

$$\begin{array}{lllllll} 0. & z & -2x_1 & \overset{\Downarrow}{-3x_2} & & & = 0 \\ 1. & & x_1 & +2x_2 & +s_1 & & = 6 \\ 2 & & 2x_1 & +x_2 & & +s_2 & = 8 \end{array}$$

If we keep increasing x_2 without limit, we might make one of the currently basic variable negative, i.e., infeasible.

$$\text{Row 1: } \quad 2x_2 + s_1 = 6 \quad \Rightarrow \quad s_1 = 6 - 2x_2$$

$$\text{Row 2: } \quad x_2 + s_2 = 8 \quad \Rightarrow \quad s_2 = 8 - x_2$$

To keep $s_1 \geq 0$, we cannot increase x_2 beyond $6/2 = 3$, i.e., $x_2 \leq 3$

Similarly, to keep $s_2 \geqslant 0$, $x_2 \leq 8$.

Choosing the smaller of the two limits, we get $x_2 \leq 3$.

On the other hand, if the dependence of s_1 on x_2 were specified as $s_1 = 6 + 2x_2$, for instance, there will be no limit placed on the value of x_2 in this case. Similarly, if the value of s_2 did not depend on x_2, e.g., $s_2 = 8$, we would not get an upper bound on x_2.

We formalize these observations into the minimum ratio test (min ratio test, in short) for picking which variable leaves the basis.

Minimum Ratio Test (min-ratio test)

For each constraint row that has a **positive coefficient** for the entering variable, compute the ratio

$$\frac{\text{right-hand side of row}}{\text{coefficient of entering var in row}}.$$

The smallest among all these ratios is the largest value the entering variable can take.

Here: Row 1: $6/2 = 3$
Row 2: $8/1 = 8$ $\Big\}$ min-ratio $= 3$.

The variable that is basic (or canonical) in the row that is the winner of the min-ratio test is the **leaving variable**.

Here, s_1 leaves the basis.

Step 5 Make entering variable basic (or canonical) in the row that won the min-ratio test using EROs.

Here, make x_2 basic in Row 1, i.e., make coefficient of x_2 in Row 1 = 1, and 0 in other rows (including Row-0).

$$\begin{array}{lrcl}
0. & z - 2x_1 \overset{\Downarrow}{-3x_2} & = 0 & \\
1. & x_1 + \boxed{2x_2} + s_1 & = 6 & 6/2 = 3 \\
2. & 2x_1 + x_2 + s_2 & = 8 & 8/1 = 8 \\
\hline
& z \overset{\Downarrow}{-\tfrac{1}{2}x_1} + \tfrac{3}{2}s_1 & = 9 & \\
& \tfrac{1}{2}x_1 + x_2 + \tfrac{1}{2}s_1 & = 3 & 3/(1/2) = 6 \\
& \boxed{\tfrac{3}{2}x_1} - \tfrac{1}{2}s_1 + s_2 & = 5 & 5/(3/2) = 10/3 \\
\hline
& z + \tfrac{4}{3}s_1 + \tfrac{1}{3}s_2 & = 32/3 & \\
& x_2 + \tfrac{2}{3}s_1 - \tfrac{1}{3}s_2 & = 4/3 & \\
& x_1 - \tfrac{1}{3}s_1 + \tfrac{2}{3}s_2 & = 10/3 & \\
\hline
\end{array}$$

$R_1 \times (\frac{1}{2})$, then $R_0 + 3R_1$, $R_2 - R_1$

$BV = \{z, x_2, s_2\}$

x_1 enters

$R_2(2/3)$, $R_0 + \frac{1}{2}R_2$, $R_1 - \frac{1}{2}R_2$

we perform all steps of the next iteration here

$BV = \{z,, x_1, x_2\}$ is optimal, as $z = 32/3 - 4/3 s_1 - 1/3 s_2$, and increasing either s_1 or s_2 from 0 will decrease z.

We performed two iterations of the simplex method above.

Consider the following LP:

$$\max\ z = 2x_1 - x_2 + x_3$$

$$\text{s.t.}\quad 3x_1 + x_2 + x_3 \le 60 \quad s_1$$

$$x_1 - x_2 + 2x_3 \le 10 \quad s_2$$

$$x_1 + x_2 - x_3 \le 20 \quad s_3$$

$$x_1, x_2, x_3 \ge 0$$

We can represent all the numbers in a compact table format, called the simplex tableau (pronounced "tablo"). All calculations are also efficiently represented in this format. This version of the simplex method is called the **tableau simplex method**.

Each tableau corresponds to a bfs, assuming it is constructed correctly. In fact, we could directly go to the starting tableau from the given LP.

$$\max\ z = 2x_1 - x_2 + x_3$$

$$\text{s.t.}\quad 3x_1 + x_2 + x_3 \le 60 \quad s_1$$

$$x_1 - x_2 + 2x_3 \le 10 \quad s_2$$

$$x_1 + x_2 - x_3 \le 20 \quad s_3$$

$$x_1, x_2, x_3 \ge 0$$

starting tableau

$z - 2x_1 + x_2 - x_3 = 0$

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs
1	−2	1	−1	0	0	0	0
0	3	1	1	1	0	0	60
0	$\boxed{1}$	−1	2	0	1	0	10
0	1	1	−1	0	0	1	20
1	0	−1	3	0	2	0	20
0	0	4	−5	1	−3	0	30
0	1	−1	2	0	1	0	10
0	0	$\boxed{2}$	−3	0	−1	1	10
1	0	0	3/2	0	3/2	1/2	25
0	0	0	1	1	−1	−2	10
0	1	0	1/2	0	1/2	1/2	15
0	0	1	−3/2	0	−1/2	1/2	5

x_1 enters

s_2 leaves

pivots

$R_0 + 2R_2$

$R_1 - 3R_2$

$R_3 - R_2$

$R_3(\frac{1}{2})$, then

$R_0 + R_3$, $R_1 - 4R_3$,

$R_2 + R_3$

all #'s (under variables) in Row-0 are ≥ 0

$\Longrightarrow$ tableau (i.e., bfs) is optimal!

optimal z-value is given as z^*.

The optimal solution is $x_1 = 15$, $x_2 = 5$, $s_1 = 10$, and $z^* = 25$.

Current bfs is optimal (for a max LP) if the numbers for each variable in Row-0 of the simplex tableau are nonnegative.

Let us recall the idea of the min ratio test, explaining it on the first tableau. Here, $BV = \{s_1, s_2, s_3\}$, $NBV = \{x_1, x_2, x_3\}$. Increasing x_1 or x_3 (from zero) will increase the z-value. We pick x_1, as the rate of increase is higher. Thus, x_1 is the entering variable.

Our goal is to move to an adjacent bfs at which the z-value is better (larger for a max LP). To move to an adjacent bfs, we exchange one basic variable with a current nonbasic variable. Here, we are going to include x_1 in the basis, and remove one of the current basic variables from the BV set. The min-ratio test helps us to identify the leaving variable.

The 3 constraint equations in the first tableau read as follows.

$$\left.\begin{aligned} 3x_1 + s_1 \qquad &= 60 \\ x_1 \quad + s_2 \quad &= 10 \\ x_1 \qquad + s_3 &= 20 \end{aligned}\right\} \Rightarrow \begin{aligned} s_1 &= 60 - 3x_1 \\ s_2 &= 10 - x_1 \\ s_3 &= 20 - x_1 \end{aligned}$$

We need to keep $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$ for feasibility. Hence we get $60 - 3x_1 \geq 0$, $10 - x_1 \geq 0$, $20 - x_1 \geq 0$, or equivalently,

$x_1 \leq 60/3$, $x_1 \leq 10$, $x_1 \leq 20$, which all hold when $x_1 \leq 10$.

When $x_1 > 10$, s_2 becomes negative, i.e., we are no longer feasible. So $x_1 = \frac{10}{1} = 10$ is the winner of the min ratio test, and since this ratio comes from Row 2, in which s_2 is canonical at present, the entering variable x_1 replaces s_2 from BV set (i.e., s_2 leaves the basis).

Notice that if we had $s_2 = 10 + x_1$ (instead of $-$), then increasing x_1 would not affect the nonnegativity of s_2. This is the reason why we do not consider rows for the min ratio test that have negative (or zero) coefficients for the entering variable.

MATH 364: Lecture 12 (09/27/2024)

Today:
- simplex for min LPs
- alternative optimal solutions in simplex method
- unbounded LPs
- big-M simplex method

Simplex method for min LPs

The criteria to decide entering variable and optimality of the bfs are opposite to those used in a max LP.

- Current bfs is optimal if all numbers in Row-0 for variables are ≤ 0 (non-positive).
- Nonbasic variable with the largest positive number in Row-0 enters (default rule for entering variable).
- min-ratio test: same as in max LP.

$$
\begin{array}{llr}
\min & z = 4x_1 - x_2 & \\
\text{s.t.} & 2x_1 + x_2 \leq 8 & s_1 \\
 & x_2 \leq 5 & s_2 \\
 & x_1 - x_2 \leq 4 & s_3 \\
 & x_1, x_2 \geq 0 &
\end{array}
$$

BV	z	x_1	x_2	s_1	s_2	s_3	rhs
	1	−4	1	0	0	0	0
s_1	0	2	1	1	0	0	8
s_2	0	0	①	0	1	0	5
s_3	0	1	−1	0	0	1	4
	1	−4	0	0	−1	0	−5
s_1	0	2	0	1	−1	0	3
x_2	0	0	1	0	1	0	5
s_3	0	1	0	0	1	1	9

Current tableau is optimal, as all #'s in Row-0 under variables are non-positive. Optimal solution is

$x_2 = 5$, $s_1 = 3$, $s_3 = 9$, and $z^* = -5$.

Another approach for min-LPs

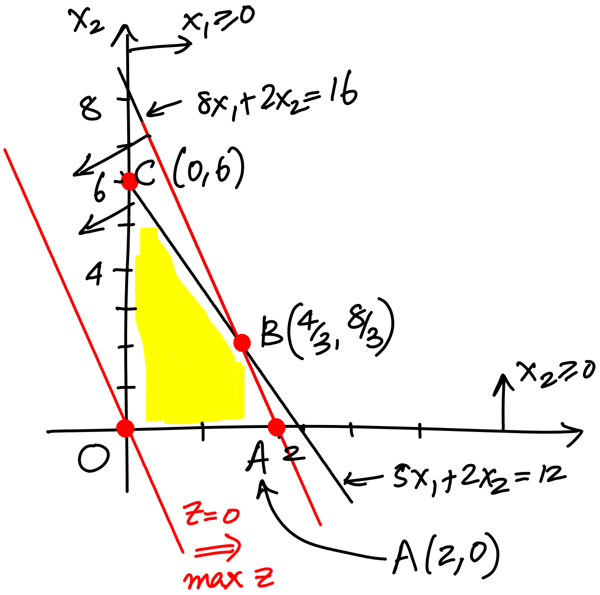
Instead of solving $\left\{\begin{array}{l}\min\ \bar{c}^T\bar{x} \\ \text{s.t. } A\bar{x}\leq\bar{b} \\ \qquad \bar{x}\geq\bar{0}\end{array}\right\}$, solve $\left\{\begin{array}{l}\max\ -\bar{c}^T\bar{x} \\ \text{s.t.}\quad A\bar{x}\leq\bar{b} \\ \qquad \bar{x}\geq\bar{0}\end{array}\right\}$

using the criteria for max-LP. Set z^*_{min} as $-z^*_{max}$, where z^*_{max} is the z^* for the max-LP. The optimal $\bar{x}$ remains same.

Alternative Optimal Solutions

Recall LP from Lecture 5:

$$\begin{array}{ll}\max & z = 4x_1 + x_2 \\ \text{s.t.} & 8x_1 + 2x_2 \leq 16 \\ & 5x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0\end{array}$$



Both A and B, as well as any point on $\overline{AB}$ are optimal solutions.

In 3D, we could have 3 or more vertices which are all optimal at the same time, and the "side" defined by all of them constitute the (infinite number of) alternative optimal solutions (similar to segment $\overline{AB}$ here)

We will have more than one optimal tableau, corresponding to each optimal bfs.

$$\max\ z = x_1 + x_2$$

s.t.

$$x_1 + x_2 + x_3 \le 1$$
$$x_1 + 2x_3 \le 1$$
$$x_1, x_2, x_2 \ge 0$$

break ties arbitrarily

BV	z	x_1	x_2	x_3	s_1	s_2	rhs	
	1	−1	−1	0	0	0	0	
s_1	0	①	1	1	1	0	1	
s_2	0	1	0	2	0	1	1	
	1	0	0	1	1	0	1	optimal tableau
x_1	0	1	①	1	1	0	1	
s_2	0	0	−1	1	−1	1	0	
	1	0	0	1	1	0	1	optimal tableau
x_2	0	1	1	1	1	0	1	
s_2	0	①	0	2	0	1	1	
	1	0	0	1	1	0	1	optimal tableau
x_2	0	0	1	−1	1	−1	0	
x_1	0	1	0	2	0	1	1	

In the last tableau, s_2 has coefficiant zero in Row-0, and could enter the basis. But we'll get back the previous optimal tableau.

Criterion:

If the coefficient of a non-basic variable in Row-0 of an optimal tableau is zero, there exist alternative optimal solutions. If we can pivot this variable into the basis, then there are alternative optimal bfs's.

There are 3 optimal bfs's here, corresponding to

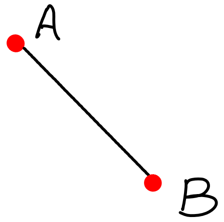
$x_1 = 1$, $x_2 = 0$, $\quad x_2 = 1$, $s_2 = 1$, and $\quad x_1 = 1$, $s_2 = 0$.

But in terms of $\underbrace{\{x_1, x_2, x_3\}}$, these 3 bfs's correspond to two optimal solutions

original variables

$$A = \begin{array}{c|c} x_1 & 1 \\ x_2 & 0 \\ x_3 & 0 \end{array} \quad \text{and} \quad B = \begin{array}{c|c} x_1 & 0 \\ x_2 & 1 \\ x_3 & 0 \end{array}.$$

Also, any point on the line segment $\overline{AB}$ is optimal, i.e., any $\bar{x} = \alpha A + (1-\alpha) B$, $0 \le \alpha \le 1$ is optimal.



$$\bar{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{array}{c|c} x_1 & \alpha \\ x_2 & 1-\alpha \\ x_3 & 0 \end{array}, \quad 0 \le \alpha \le 1.$$

↳ this expression is analogous to the parametric vector form of solutions to $A\bar{x} = \bar{b}$, when there are free variables.

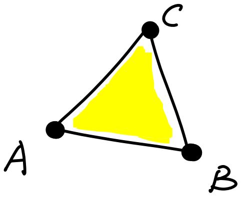
For instance, $\alpha = 0.5$, we get the mid point of $\overline{AB}$.

Indeed, $z = x_1 + x_2 = \alpha + 1 - \alpha = 1 = z^*$ for any such α.

With 3 different optimal vertices A, B, C, all optimal solutions can be written as

$$\bar{x} = \alpha_A A + \alpha_B B + \alpha_C C \quad , \quad 0 \le \alpha_A, \alpha_B, \alpha_C \le 1$$
$$\alpha_A + \alpha_B + \alpha_C = 1$$

e.g., $\alpha_A = \frac{1}{2}, \alpha_C = \frac{1}{2}, \alpha_B = 0$, gives the mid point of '$\overline{AC}$'.

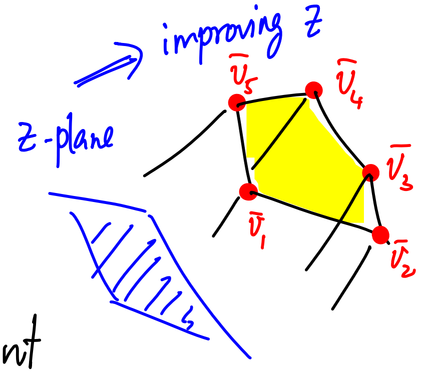


$\bar{x}$ here is a **convex combination** of A, B, C.

A linear combination is $\bar{x} = \alpha_A A + \alpha_B B + \alpha_C C$, for $\alpha_A, \alpha_B, \alpha_C \in \mathbb{R}$.

Thus, a convex combination is a special linear combination.

Idea in 3D (and higher dimensions): Example



The z-plane hits flush against an entire face, here shown with five corner points $\bar v_j$, $j=1$–5, for instance

Each corner point $\bar v_j$ is optimal, and so is any point in the shaded region. Any point in the pentagon is a convex combination of the $\bar v_j$s. → more generally, there could be many $\bar v_j$'s (not just 5).

Def A **convex combination** of $\bar v_1, \ldots, \bar v_n$ is

$$\bar x = \sum_{j=1}^{n} \alpha_j \bar v_j, \quad 0 \le \alpha_j \le 1, \quad \sum_{j=1}^{n} \alpha_j = 1.$$

For instance, when $\alpha_2 = 1$, $\alpha_j = 0$ for $j = 1,3,4,5$, $\bar x = \bar v_2$. Similarly, when $\alpha_3 = \alpha_5 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \alpha_4 = 0$, we get $\bar x = \frac{1}{2}(\bar v_3 + \bar v_5)$, which is the midpoint of the line segment connecting $\bar v_3$ and $\bar v_5$. And when $\alpha_j = \frac{1}{5}$ for all j, $\bar x$ is the "centroid" (or average) of all the corner points.

Unbounded LPs

Recall that in 2D, when you could slide the z-line without limits while improving z and remaining feasible, the LP is unbounded.

$$\max\ z = 2x_2$$
$$\text{s.t.}\quad x_1 - x_2 \leq 4$$
$$-x_1 + x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

BV	z	x_1	x_2	s_1	s_2	rhs
	1	0	−2	0	0	0
s_1	0	1	−1	1	0	4
s_2	0	−1	1	0	1	1
	1	−2	0	0	2	2
s_1	0	0	0	1	1	5
x_2	0	−1	1	0	1	1

We do not have any candidates for the min-ratio test in the second tableau ⇒ LP is unbounded. x_1 could enter the basis and improve the z-value, but there is no limit on how much the increase can be.

The equations (in Rows 1 & 2) are

$$s_1 = 5$$
$$-x_1 + x_2 = 1 \Rightarrow x_2 = 1 + x_1$$

} as x_1 increases, both s_1 and x_2 stay > 0.

Thus we could keep increasing x_1, and hence improving z, without ever ever encountering infeasibility. Hence the LP is unbounded!

Criterion: The tableau has a non-basic variable that could enter and improve the value of z, but there are no candidates for min-ratio.

→ the coefficient cannot be zero

So far, the LPs we have looked at are all of the form

$$\left\{ \begin{array}{l} \max/\min\ \bar{c}^T\bar{x} \\ \quad A\bar{x} \le \bar{b} \\ \quad \bar{x} \ge \bar{0} \end{array} \right\}$$

where $\bar{b} \ge \bar{0}$. $\bar{x} = \bar{0}$ is always feasible here.

So, we do not get infeasible LPs.

To consider infeasible LPs, we introduce a general tableau simplex method that could handle $\ge$ and $=$ constraints.

The big-M Method of Tableau Simplex

Can handle $\ge$ or $=$ constraints

IDEA
- * add artificial variables in order to obtain a starting bfs.
- * modify objective function so as to force the artificial variables to zero in the optimal solution.

$$\begin{array}{ll} \min & z = 2x_1 + 3x_2 \\ \text{s.t.} & 2x_1 + x_2 \ge 4 \text{ ——— } (1) \\ & x_1 - x_2 \ge -1 \text{ ——— } (2) \\ & x_1, x_2 \ge 0 \end{array}$$

Step 1 Modify any constraints so that all rhs values are nonnegative.

Recall that we can read off the bfs from the tableau – assuming all rhs values are ≥ 0. Else, feasibility is violated.

If the rhs value of a constraint is negative, scale it by -1. The sense of the inequality is reversed here.

$$(2) \times -1 \Rightarrow -1(x_1 - x_2 \ge -1) \qquad -x_1 + x_2 \le 1 \text{ ——— } (2')$$

For instance, consider $-3 \ge -5$. Multiplying this inequality by -1 indeed reverses the sense of the inequality: $-(-3 \ge -5) \Rightarrow 3 \le 5$.

One advantage of using slack variables is that we can choose the obvious starting bfs by picking the slack variables in the BV. But for '$\geq$' constraints, we subtract excess variables, which are not canonical. Similarly, we do not have obvious canonical variables for '$=$' constraints. Hence, we add artificial variables for such constraints.

Step 2 Add an artificial variable a_i to constraint i if it is a $\geq$ or $=$ constraint, and add $a_i \geq 0$.

$$(1) \Rightarrow \quad 2x_1 + x_2 + a_1 \geq 4 \text{ ——— } (1')$$

Step 3 For max-LP, add $-Ma_i$ to the objective function (z); and for min-LP, add $+Ma_i$ to z, where M is a large positive number.

$$\begin{aligned} \min \quad & z = 2x_1 + 3x_2 + Ma_1 \\ \text{s.t.} \quad & 2x_1 + x_2 + a_1 \geq 4 \text{ —— } (1') \\ & -x_1 + x_2 \leq 1 \text{ ——— } (2') \\ & x_1, x_2, a_1 \geq 0 \end{aligned}$$

→ this term forces a_1 to zero in any optimal solution, assuming the LP is not infeasible. With the M coefficient, as long as $a_1 > 0$, z is very huge due to the Ma_1 term, however small $a_1 > 0$ is.

M acts like ∞, but we can "handle" it!

So $\quad 3M + 1 > 2M + 123456$

$$-2M + 10 < -M - 2500000$$

Step 4 Convert all inequalities to standard form (using slack/excess vars).

$$\begin{aligned} \min \quad & z = 2x_1 + 3x_2 + Ma_1 \\ \text{s.t.} \quad & 2x_1 + x_2 + a_1 - e_1 = 4 \quad \text{——} \ (1') \\ & -x_1 + x_2 + s_2 = 1 \quad \text{——} \ (2') \\ & x_1, x_2, a_1, e_1, s_2 \geq 0 \end{aligned}$$

We will describe the remaining steps in the next lecture...

MATH 364: Lecture 13 (10/01/2024)

Next Tuesday (Oct 8): Midterm (in-class; practice midterm is posted)
Topics: Everything before big-M method

Today:
- infeasibility in tableau simplex
- urs vars

We finish describing the steps of the big-M simplex method.
Recall steps 1-4 from Lecture 12...

Step 5 Convert LP to canonical form by converting the coefficients of artificial variables a_i in Row-0 to zero (using EROs). The initial bfs will then have all slack (s_i) and all artificial variables (a_i's). Solve the resulting LP tableau using regular simplex method.

BV	z	x_1	x_2	e_1	s_2	a_1	rhs
	1	-2	-3	0	0	$-M$	0
	0	2	1	-1	0	1	4
	0	-1	1	0	1	0	1
	1	$2M-2$	$M-3$	$-M$	0	0	$4M$
a_1	0	②	1	-1	0	1	4
s_2	0	-1	1	0	1	0	1
	1	0	-2	-1	0	$-(M-1)$	4
x_1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	2
s_2	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	3

$R_0 + MR_1$

$R_0 - (2M-2)R_1$ ← new R_1

$\cancel{M}-3-(2\cancel{M}-2)\frac{1}{2}$

$-\cancel{M}+(2\cancel{M}-2)\frac{1}{2}$

$4\cancel{M}-(2\cancel{M}-2)2$

Optimal solution: $x_1=2$, $s_2=3$, $z^*=4$.

If any a_i is >0 in the optimal tableau (i.e., it is basic), the original LP is infeasible.

Detecting infeasible LPs

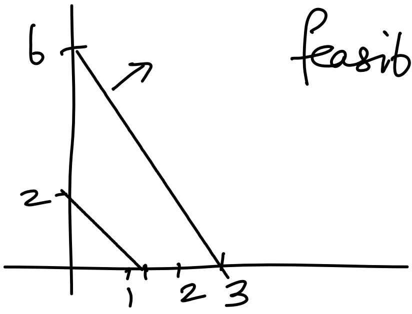
Recall that if all constraints are $\leq$, all rhs values (b_i's) are ≥ 0, then $\bar{x}=\bar{0}$ is feasible. But in more general settings, we can detect infeasible LPs using the big-M simplex method.

Criterion

If an artificial var is basic in the optimal tableau (i.e., is >0), then the original LP is infeasible.

$$\min\ z = 3x_1 + Ma_1 + Ma_2$$

$$\begin{aligned} \text{s.t.}\quad & 2x_1 + x_2 + a_1 \geq 6 \quad e_1 \\ & 3x_1 + 2x_2 + a_2 = 4 \\ & x_1, x_2, a_1, a_2, e_1 \geq 0 \end{aligned}$$



feasible region is empty

BV	z	x_1	x_2	e_1	a_1	a_2	rhs
	1	-3	0	0	$-M$	$-M$	0
	0	2	1	-1	1	0	6
	0	3	2	0	0	1	4
	1	$5M-3$	$3M$	$-M$	0	0	$10M$
a_1	0	2	1	-1	1	0	6
a_2	0	③	2	0	0	1	4
	1	0	$-M/3+2$	$-M$	0	$-5/3M+1$	$10/3M+4$
a_1	0	0	$-1/3$	-1	1	$-2/3$	$10/3$
x_1	0	1	$2/3$	0	0	$1/3$	$4/3$

$R_0 + MR_1 + MR_2$

→ optimal !

$R_0 - (5M-3)R_2 \to$ new R_2

$3M - (5M-3)\frac{2}{3}$

$10M - (5M-3)4/3$

Since $a_1 = 10/3$ in the optimal tableau, the original LP is infeasible.

Unrestricted in Sign (Urs) Variables

* If x_i is urs, replace x_i by $x_i^+ - x_i^-$ in all constraints and in the objective function, and add $x_i^+, x_i^- \geq 0$.

IDEA : $x_i = x_i^+ - x_i^- \quad (x_i^+, x_i^- \geq 0)$

If $x_i = 5$, $x_i^+ = 5$, $x_i^- = 0$ works, and

if $x_i = -3$, $x_i^+ = 0$, $x_i^- = 3$ works.

→ $x_i^+ = 8$, $x_i^- = 3$ works too, but we will show only one of x_i^+, x_i^- can be basic in the optimal tableau.

$$\max\ z = 2x_1 + \overset{\to\, x_2^+ - x_2^-}{x_2}$$

s.t.

$$3x_1 + x_2 \leq 6$$
$$x_1 + x_2 \leq 4$$
$$x_1 \geq 0,\ x_2 \text{ urs}$$

$$\max\ z = 2x_1 + x_2^+ - x_2^-$$

s.t.

$$3x_1 + x_2^+ - x_2^- \leq 6 \quad s_1$$
$$x_1 + x_2^+ - x_2^- \leq 4 \quad s_2$$
$$x_1, x_2^+, x_2^- \geq 0$$

BV	z	x_1	x_2^+	x_2^-	s_1	s_2	rhs
	1	-2	-1	1	0	0	0
s_1	0	$\boxed{3}$	1	-1	1	0	6
s_2	0	1	1	-1	0	1	4
	1	0	$-1/3$	$1/3$	$2/3$	0	4
x_1	0	1	$1/3$	$-1/3$	$1/3$	0	2
s_2	0	0	$\boxed{2/3}$	$-2/3$	$-1/3$	1	2
	1	0	0	0	$1/2$	$1/2$	5
x_1	0	1	0	0	$1/2$	$-1/2$	1
x_2^+	0	0	1	-1	$-1/2$	$3/2$	3

optimal solution $x_1 = 1$, $x_2 = x_2^+ - x_2^- = 3 - 0 = 3$, $z^* = 5$.

Note that the columns of x_2^+ and x_2^- are -1 multiples of each other. Hence both cannot be basic in a tableau, and we get x_2 modeled correctly.

If $x_i \leq 0$ to start with, replace x_i by $-x_i'$ everywhere, and add $x_i' \geq 0$. In the end set $x_i = -x_i'$ (in optimal solution).

→ instead of it being urs

Putting it all together

$$
\begin{aligned}
\min\ & z = 2x_1 - 3x_2 \\
\text{s.t.}\ & x_1 + 3x_2 \leq 9 \\
& 2x_1 + 5x_2 \geq -6 \\
& x_2 \geq 1 \\
& x_1 \text{ urs},\ x_2 \geq 0
\end{aligned}
$$

Solve this LP using the big-M method.

We outline the steps of big-M simplex method.

Step 1 Scale any constraint with < 0 rhs

$$-(2x_1 + 5x_2 \geq -6) \Rightarrow \quad -2x_1 - 5x_2 \leq 6$$

Steps 2&3

Add artificial var a_i to constraint i if it is $\geq$ or $=$; and add $a_i \geq 0$.

add $\pm M a_i$ to z (obj. fn) ($+Ma_i$ for min LP).

$$
\begin{aligned}
\min\ & z = 2x_1 - 3x_2 + Ma_3 \\
\text{s.t.}\ & x_1 + 3x_2 \leq 9 \quad s_1 \\
& -2x_1 - 5x_2 \leq 6 \quad s_2 \\
& x_2 + a_3 \geq 1 \quad e_3 \\
& x_1 \text{ urs},\ x_2 \geq 0,\ a_3 \geq 0
\end{aligned}
$$

Step 4 Replace urs variable x_i by $x_i^+ - x_i^-$, add $x_i^+, x_i^- \geqslant 0$. Replace x_i by $-x_i^-$ when $x_i \leq 0$, and add $x_i^- \geqslant 0$.

$$\begin{array}{lrl}
\min & Z = & 2x_1^+ - 2x_1^- - 3x_2 + Ma_3 \\
\text{s.t.} & & x_1^+ - x_1^- + 3x_2 \leq 9 \quad s_1 \\
& & -2x_1^+ + 2x_1^- - 5x_2 \leq 6 \quad s_2 \\
& & x_2 + a_3 \geqslant 1 \quad e_3 \\
& & x_1^+, x_1^-, \quad x_2 \geqslant 0, a_3 \geqslant 0
\end{array}$$

Step 5 Convert LP to standard form using slack/excess variables.

$$\begin{array}{lrl}
\min & Z = & 2x_1^+ - 2x_1^- - 3x_2 + Ma_3 \\
\text{s.t.} & & x_1^+ - x_1^- + 3x_2 + s_1 = 9 \\
& & -2x_1^+ + 2x_1^- - 5x_2 + s_2 = 6 \\
& & x_2 - e_3 + a_3 = 1 \\
& & x_1^+, x_1^-, \quad x_2 \geqslant 0, a_3 \geqslant 0, s_1, s_2, e_3 \geqslant 0
\end{array}$$

Step 6 Use slack and artificial vars in the starting bfs, convert tableau to canonical form.

Proceed with subsequent steps of tableau simplex method.

13.6

$$\min\ z = 2x_1^+ - 2x_1^- - 3x_2 + Ma_3$$

s.t.

$$x_1^+ - x_1^- + 3x_2 + s_1 = 9$$

$$-2x_1^+ + 2x_1^- - 5x_2 + s_2 = 6$$

$$x_2 - e_3 + a_3 = 1$$

all vars ≥ 0

BV	z	x_1^+	x_1^-	x_2	s_1	s_2	e_3	a_3	rhs	
	1	-2	2	3	0	0	0	-M	0	$R_0 + MR_3$
	0	1	-1	3	1	0	0	0	9	
	0	-2	2	-5	0	1	0	0	6	
	0	0	0	1	0	0	-1	1	1	
	1	-2	2	M+3	0	0	-M	0	M	
s_1	0	1	-1	3	1	0	0	0	9	
s_2	0	-2	2	-5	0	1	0	0	6	
a_3	0	0	0	1	0	0	-1	1	1	
	1	-2	2	0	0	0	3	-(M+3)	-3	$R_0 - (M+3)R_3$
s_1	0	1	-1	0	1	0	3	-3	6	
s_2	0	-2	2	0	0	1	-5	5	11	
x_2	0	0	0	1	0	0	-1	1	1	
	1	-3	3	0	-1	0	0	-M	-9	
e_3	0	1/3	-1/3	0	1/3	0	1	-1	2	
s_2	0	-1/3	1/3	0	5/3	1	0	0	21	
x_2	0	1/3	-1/3	1	1/3	0	0	0	3	
	1	0	0	0	-16	-9	0	-M	-198	
e_3	0	0	0	0	2	1	1	-1	23	
x_1^-	0	-1	1	0	5	3	0	0	63	
x_2	0	0	0	1	2	1	0	0	24	

optimal solution: $x_1 = x_1^+ - x_1^- = 0 - 63 = -63$, $x_2 = 24$, $e_3 = 23$; $z^* = -198$.

MATH 364: Lecture 14 (10/03/2024)

Today: * Review for midterm

Optimal Solutions and # Optimal BFS's

Recall bfs ≡ corner point

We note the following points:

- * an **optimal solution** is any feasible point that is optimal; it may or may ~~may~~ not be a corner point, i.e., it may or may not be a bfs.
- * But if the LP has a unique optimal solution, then that optimal solution must be a bfs.
- * If the LP has alternative optimal solutions (Case 2), it must have **infinitely many optimal solutions**.
- * The total # bfs's possible is finite, since we can have at most $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ choices for the set of basic variables, but not all of them may give a bfs. Hence the **# optimal bfs's** is also **finite**.

Hw6, Problem 5

You will get 3 optimal bfs's. Describe them as 4-vectors in the form $\bar{u} = \begin{array}{c|c} x_1 & \cdot \\ x_2 & \cdot \\ x_3 & \cdot \\ x_4 & \cdot \end{array}$, $\bar{v} = \begin{array}{c|c} x_1 & \cdot \\ x_2 & \cdot \\ x_3 & \cdot \\ x_4 & \cdot \end{array}$, and $\bar{w} = \begin{array}{c|c} x_1 & \cdot \\ x_2 & \cdot \\ x_3 & \cdot \\ x_4 & \cdot \end{array}$, say.

Then describe all optimal solutions as convex combinations of $\bar{u}$, $\bar{v}$, and $\bar{w}$.

Hw6, Problem 3

Alternative Optimal solutions exist if there is a non-basic variable with coefficient 0 in Row-0.

Here, x_1 is non-basic with 0 in its Row-0.

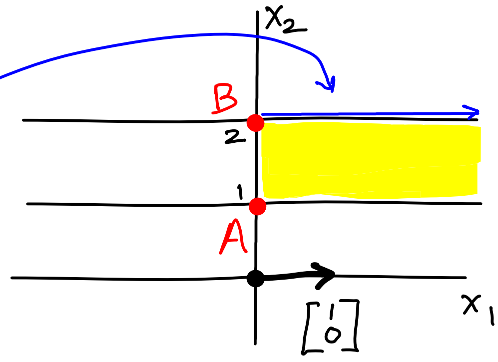
$\Rightarrow$ alternative optimal solutions exist.

BV	z	x_1	x_2	x_3	x_4	rhs
	1	0	0	0	4	5
x_3	0	−2	0	1	2	3
x_2	0	−3	1	0	1	1

But there are no candidates for min ratio test $\Rightarrow$ x_1 cannot enter the basis. So, there are no alternative optimal bfs's.

$z^* = 2$ here.

Consider $\left\{ \begin{array}{l} \max\ x_2 \\ \text{s.t.}\ 1 \le x_2 \le 2 \\ \quad x_1, x_2 \geq 0 \end{array} \right\}$.



There is one optimal bfs at B(0,2) and infinitely many optimal solutions.

All optimal solutions can be given as $\bar{x} = \begin{bmatrix}0\\2\end{bmatrix} + \alpha \begin{bmatrix}1\\0\end{bmatrix}$, $\alpha \geqslant 0$, i.e.,

$$\bar{x} = \begin{bmatrix}\alpha\\2\end{bmatrix}, \alpha \geqslant 0.$$

Hw 6, Problem 4

What happens if we were to pivot x_1 in?

We come to the conclusion that the LP is unbounded after that one pivot.

	z	x_1	x_2	x_3	x_4	rhs
	1	-5	0	0	-1	5
x_3	0	2	0	1	0	4
x_2	0	3	1	0	-2	6
	1	0	0	$5/2$	-1	15
x_1	0	1	0	$1/2$	0	2
x_2	0	0	1	$-3/2$	-2	0

LP is unbounded

But we can get to that conclusion without pivoting x_1 in!

Problems from Practice Midterm

1.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	c_1	c_2	0	$c_3=0$	c_4	$c_5=0$	z^*
0	3	a_1	1	0	a_2	$a_3=0$	1
0	-1	-2	0	$a_4=1$	-1	$a_5=0$	$b\geq 0$
0	a_6	-4	0	0	-3	$a_7=1$	3

x_1, x_2, x_5 cannot be basic (they must be unit vector columns).

So, x_3, x_4, x_6 must be basic.
x_3 is basic in Row-1.

$c_3=0$, $a_4=1$, $c_5=0$, $a_3=0$, $a_5=0$, $a_7=1$ and $b\geq 0$ always hold.

x_4 coefficient in Row 3 is $0 \Rightarrow x_4$ is basic in Row 2, and hence x_6 is basic in Row 3.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	$c_1 = 0$	$c_2 = 0$	0	$c_3 = 0$	$c_4 = 0$	$c_5 = 0$	z^*
0	3	$a_1 > 0$	1	0	$a_2 > 0$	$a_3 = 0$	1
0	-1	-2	0	$a_4 = 1$	-1	$a_5 = 0$	$b \geq 0$
0	a_6	-4	0	0	-3	$a_7 = 1$	3

(a) The current solution is optimal, and there are alternative optimal basic feasible solutions.

optimality: $c_1 \leq 0, c_2 \leq 0, c_4 \leq 0$

For alternative optimal bfs, we should be able to pivot a non-basic var with Row-0 coeff $= 0$ into the basis.

$(c_1 = 0)$ OR $(c_2 = 0, a_1 > 0)$ OR $(c_4 = 0, a_2 > 0)$,

or any combinations of above settings.

(b) LP unbounded.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	c_1	$c_2 > 0$	0	$c_3 = 0$	$c_4 > 0$	$c_5 = 0$	z^*
0	3	$a_1 \leq 0$	1	0	$a_2 \leq 0$	$a_3 = 0$	1
0	-1	-2	0	$a_4 = 1$	-1	$a_5 = 0$	$b \geq 0$
0	a_6	-4	0	0	-3	$a_7 = 1$	3

$(c_2 > 0$ and $a_1 \leq 0)$ OR $(c_4 > 0$ and $a_2 \leq 0)$, or both.

3. Careful about sign restrictions!
Plot at least one z-line.

4.

z	x_1	x_2	x_3	s_1	s_2	rhs
1	0	−3	0	−3	−1/2	−20
0	1	−1	0	1	−1	2
0	0	2	1	0	1/2	2
1	3	−6	0	0	−7/2	−14
0	1	−1	0	1	−1	2
0	0	2	1	0	1/2	2
1	3	8	7	0	0	0
0	1	3	2	1	0	6
0	0	4	2	0	1	4

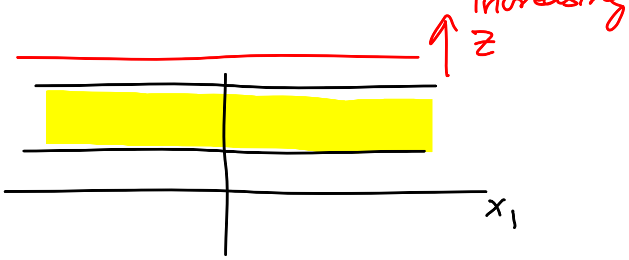
min LP, as all #s in Row-0 (under vars) are ≤ 0.

$$\min\ z = -3x_1 - 8x_2 - 7x_3$$
$$\text{s.t.}\quad x_1 + 3x_2 + 2x_3 \le 6$$
$$4x_2 + 2x_3 \le 4$$
$$x_1,\ x_2,\ x_3 \ge 0$$

5. T/F

a) feasible region has no corner points.

FALSE



$\max\ x_2$
s.t. $1 \le x_2 \le 2$
x_1 urs

has no corner points, but $z^* = 2$ (i.e., not unbounded LP)

b) False.
If min ratio $= 0$, z does not change.

MATH 364 : Lecture 16 (10/10/2024)

Today:
- more AMPL
- sensitivity analysis

Offer on midterm:

- If you get ≥92% in final, final score will replace mid-term score.
- If you get ≥85% (<92%), the weights for final will be 30%, and mid-term = 10%.

Updates:
- Hw 7 will be due Tuesday, Oct 22.
- Final exam will be take-home open book (but NO AI tools allowed).

More AMPL

Chukee problem: We could consider a generalization where there are multiple types of toys and multiple types of operations to make the toys.

Toy ↓	Assembly	Paint	. . . .
Dirty	1500	800	
Ugly	1200	700	
⋮	.		

See the course web page for details.

Sensitivity Analysis

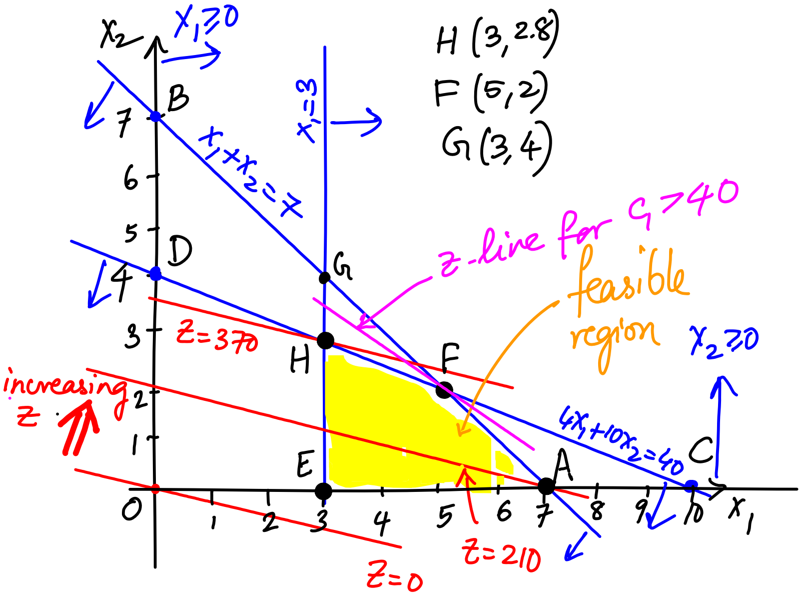
How do changes in parameters affect the optimal solution?

After solving the LP, say, you realize one of the objective function coefficients is changed by a little bit, but the rest of the problem remains the same. Could we find the new optimal solution quickly from the previous optimal solution, without re-solving the changed LP from scratch?

More generally, we want to study how sensitive the optimal solution and the optimal basis are to changes in the data of the problem. Just as we did when developing the simplex method to solve LPs, we will first study sensitivity analysis in 2D using the graphical method.

Recall: Farmer Jones LP:

$$
\begin{array}{lrll}
\max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\
\text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\
 & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\
 & 10x_1 & \geq 30 & \text{(min corn)} \\
 & x_1,\ x_2 & \geq 0 & \text{(non-negativity)}
\end{array}
$$



H(3, 2.8) is the optimal solution.

Q. For what values of revenue/acre of corn (currently 30) is the current solution H(3, 2.8) optimal?

Effect of Change in an objective function Coefficient

Say, price/bu of corn goes up to \$4 (from \$3). Should Jones still farm 3 acres of Corn and 2.8 acres of wheat?

→ So, objective function is $\max z = c_1x_1 + 100x_2$

More generally, let revenue/acre of corn = \$ c_1, for what values of c_1 is the current Solution optimal?

If objective function is $\max z = c_1x_1 + 100x_2$, the slope of z-line is $-\frac{c_1}{100}$.

When $c_1 = 40$, slope of z-line = slope of (labor hrs) line.

For $c_1 \geq 40$, F(5,2) becomes the optimal solution, until $c_1 = 100$. When $c_1 \geq 100$, A(7,0) becomes the optimal solution.

Hence H(3,2.8) is the unique optimal solution for

$$\boxed{c_1 \leq 40} \rightsquigarrow \text{assuming } c_1 \geq 0.$$

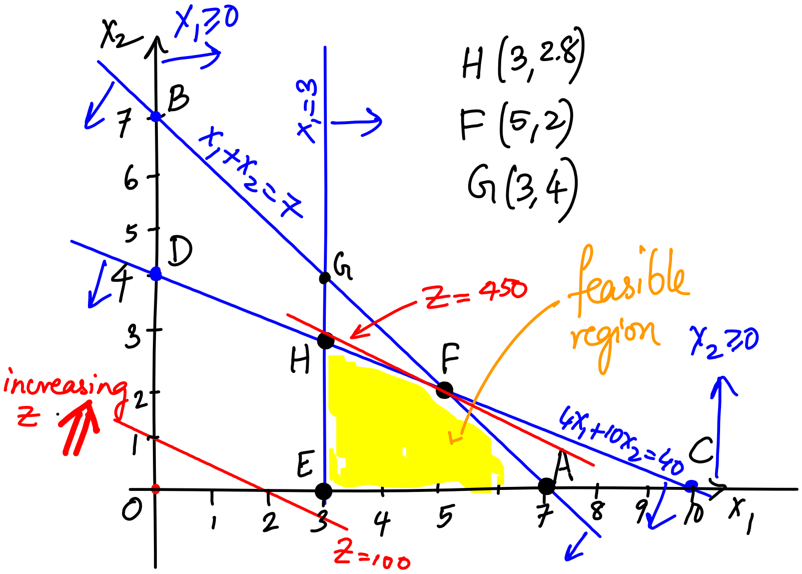
If c_1 goes negative, the slope will change sign. But since c_1 is the revenue/acre of corn, $c_1 \geq 0$ makes sense.

Changing revenue/acre of wheat

First, let's assume price/bushel of corn is \$5. So the objective function is max $z = 50x_1 + 100x_2$. Now, F(5,2) is optimal, with $z^* = 450$. The analysis becomes more interesting here, as compared to the original Farmer Jones LP.

$$\begin{array}{llll} \max & z = \overset{50}{\cancel{30}}x_1 + 100x_2 & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\ & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\ & 10x_1 & \geq 30 & \text{(min corn)} \\ & x_1,\ x_2 & \geq 0 & \text{(non-negativity)} \end{array}$$

Optimal solution is at F(5,2), with $z^* = 450$.



Q. For what values of revenue/acre of wheat (c_2; = 100 now) is the current solution F(5,2) optimal?

We'll finish this topic in the next lecture...

MATH 364: Lecture 17 (10/15/2024)

- Submit anonymous feedback @ any time!
- You're welcome to use Matlab/Python to show work for tableau simplex in homework.

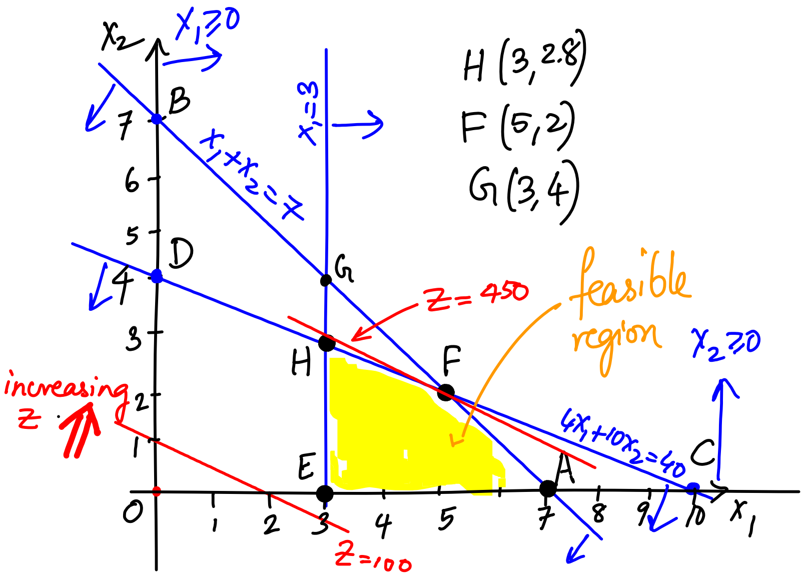
Today: * sensitivity analysis
- change in c_2 (coefficient of x_2)
- change in b_i, shadow price

Changing revenue/acre of wheat

First, let's assume price/bushel of corn is \$5. So the objective function is max $z = 50x_1 + 100x_2$. Now, $F(5,2)$ is optimal, with $z^* = 450$. The analysis becomes more interesting here, as compared to the original Farmer Jones LP.

$$
\begin{array}{llll}
\max & z = \overset{50}{\cancel{30}}x_1 + 100x_2 & & \text{(total revenue)} \\
\text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\
 & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\
 & 10x_1 & \geq 30 & \text{(min corn)} \\
 & x_1,\ x_2 & \geq 0 & \text{(non-negativity)}
\end{array}
$$

Optimal solution is at $F(5,2)$, with $z^* = 450$.



Q. For what values of revenue/acre of wheat (c_2; $=100$ now) is the current solution $F(5,2)$ optimal?

Let c_2 be the coefficient of x_2 in the objective function.

Slope of z-line is $\frac{-50}{c_2}$. $\left(z = 50x_1 + c_2 x_2\right)$

Both the (labor_hrs) and the (land_available) constraints are binding at the current optimal solution (at F(5,2)). This solution remains optimal as long as the slope of the z-line is in between the slopes of the two binding constraints at F.

Thus, current solution remains optimal as long as

$$\frac{-1}{1} \le \frac{-50}{c_2} \le \frac{-4}{10}$$

→ We invert the three fractions, and scale the two inequalities by −1, and hence the inequality senses stay as ≤.

$$\Rightarrow \quad 1 \le \frac{c_2}{50} \le \frac{10}{4} \quad \Rightarrow \quad 50 \le c_2 \le 125.$$

Equivalently, if the price per bushel of wheat is $\$w$, then for $\frac{50}{25} \le w \le \frac{125}{25}$, i.e., $2 \le w \le 5$, the current solution remains optimal.

→ # bushels/acre of wheat.

e.g., if $w = \$3.5$, $x_1 = 5$, $x_2 = 2$ is still optimal, but new

$$z^* = 50x_1 + 3.5 \times 25 \times 2 = \$425.$$

Change in RHS Coefficient (Stay with $z = 50x_1 + 100x_2$).

max $z = \overset{50}{\cancel{30}}x_1 + 100x_2$ (total revenue)

s.t.

$x_1 + x_2 \leq 7$ (land availability)

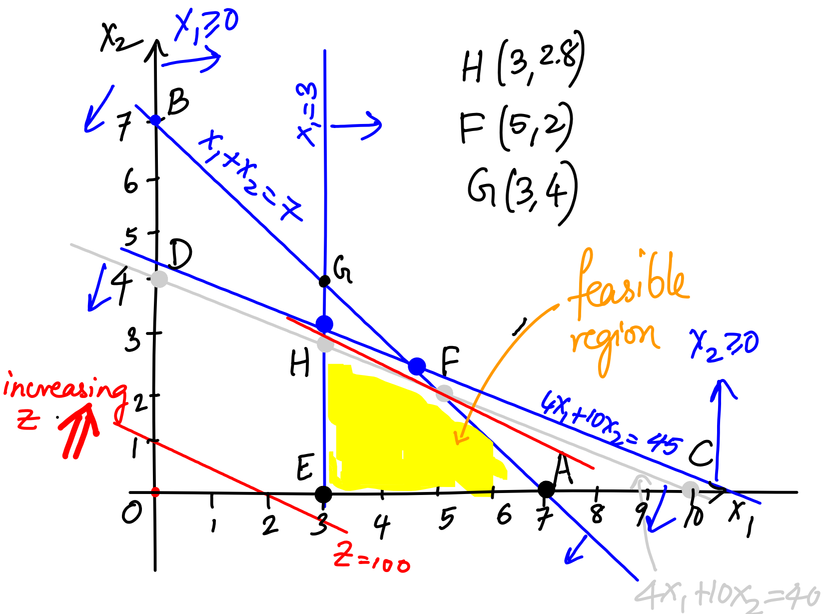
$4x_1 + 10x_2 \leq 40$ (labor hrs)

$10x_1 \geq 30$ (min corn)

$x_1, x_2 \geq 0$ (non-negativity)

Optimal solution is at $F(5,2)$,

$z^* = 450$.



Q. What happens if Jones has 45 hrs of labor/wk?

It appears the optimal basis remains same ($x_1 > 0$ and $x_2 > 0$) as long as the (labor hrs) line moves parallel to itself between A(7,0) and G(3,4).

By sliding the (labor hrs) line up/down, we see that between 28 and 52 hrs/week, the current basis remains optimal.

What is the effect of changing b_i on z^* and $\bar{x}^*$?

optimal solution

optimal objective fn. value

Let us assume b_2 (here) changes from 40 to $40 + \Delta$.

As F remains optimal for $28 \leq b_2 \leq 52$, we get

$$28 \leq 40 + \Delta \leq 52 \implies -12 \leq \Delta \leq 12.$$

range of values of Δ for which current basis remains optimal.

How did we get the limits of 28 and 52? Here are some details.

First, notice that as long as the (land) and (labor.hrs) constraints remain binding, the current basis, given by $BV = \{x_1, x_2, e_3\}$ will remain optimal, even if z^* and the values of x_1, x_2 might be different.

Notice that (min.corn) constraint is non-binding at $F(5,2)$, and hence $e_3 > 0$, making it basic.

$e_3 \rightarrow$ excess var for (min corn) constraint

In general, let the rhs value of the (labor hrs) constraint be b_2. We can ask: "For what values of b_2 is the current basis ($BV = \{x_1, x_2, e_3\}$) optimal?"

As long as $4x_1 + 10x_2 = b_2$ is (at or) below G and (at or) above A, current basis remains optimal.

At $A(7,0)$, $b_2 = 4 \times 7 + 10 \times 0 = 28$, and

at $G(3,4)$, $b_2 = 4 \times 3 + 10 \times 4 = 52$.

Hence, for $28 \le b_2 \le 52$, the current basis is optimal.

New $z^*, \bar{x}^*$?   New F, i.e., F_Δ, (and $\bar{x}^*_\Delta$)

$$x_1 + x_2 = 7 \quad \text{———}(1)$$
$$4x_1 + 10x_2 = 40+\Delta \quad \text{———}(2)$$

(2)−4(1): $6x_2 = 12+\Delta \Rightarrow x_2 = 2 + \frac{\Delta}{6}$

so, (1) $\Rightarrow x_1 = 5 - \frac{\Delta}{6}$. So $F_\Delta = \left(5-\frac{\Delta}{6}, 2+\frac{\Delta}{6}\right). \left(= \bar{x}^*_\Delta\right)$

$$\Rightarrow z^*_\Delta = 50x_1 + 100x_2 = 50\left(5-\frac{\Delta}{6}\right) + 100\left(2+\frac{\Delta}{6}\right)$$
$$= 450 + \left(\frac{50}{6}\right)\Delta = 450 + \boxed{\left(\frac{25}{3}\right)}\Delta.$$

shadow price of (labor-hrs).

Def The **shadow price** of constraint i is the amount by which the value of z improves for a unit increase in b_i (rhs) provided the optimal basis remains same after the increase.

Economic interpretation

Shadow price here is the price that Jones is willing to pay to get an extra hour of labor. For instance, if he can get someone to work 5 hrs extra for \$6/hr, he will take it, as his total revenue will increase by $\$\frac{25}{3} = \$8.33$ for each extra labor hour, resulting in a net profit.

Shadow price of land constraint?

Go back to the original problem with $4x_1+10x_2 \le 40$, but now change b_1 in $x_1+x_2 \le b_1$ so that F still remains optimal. We get $5.8 \le b_1 \le 10$ as the range.

x_1+x_2 @ $H(3, 2.8)$ $\quad$ x_1+x_2 @ $C(10,0)$

Let b_1 change from 7 to $7+\Delta$. The new optimal solution F is

$$x_1 + x_2 = 7+\Delta \text{ —— } (1)$$
$$4x_1 + 10x_2 = 40 \text{ —— } (2)$$

$$(2)-4(1) \Longrightarrow \quad 6x_2 = 12-4\Delta \quad \Rightarrow \quad x_2 = 2-\tfrac{2}{3}\Delta.$$

So, (1) $\Rightarrow$ $x_1 = 7+\Delta-(2-\tfrac{2}{3}\Delta) = 5+\tfrac{5}{3}\Delta.$

i.e., $F\left(5+\tfrac{5}{3}\Delta, 2-\tfrac{2}{3}\Delta\right)$ is the new optimal solution, and

$$z^* = 50x_1 + 100x_2 = 50\left(5+\tfrac{5}{3}\Delta\right)+100\left(2-\tfrac{2}{3}\Delta\right)$$
$$= 450 + \tfrac{50}{3}\Delta.$$

$\Rightarrow$ Shadow price of acres constraint is $\frac{50}{3}$. In other words, Jones would be willing to pay up to $\$\frac{50}{3}$ for one extra acre of land.

Shadow price of the (min corn) constraint is zero, as $30 \to 30+\Delta$ will not change the optimal solution $F(5,2)$.

Shadow price of a non-binding constraint is always zero!

MATH 364: Lecture 18 (10/17/2024)

Today:
- Tableau simplex in Matlab
- simplex method in matrix form

You can do the computations (EROs) for the tableau simplex method in Matlab (or Python). See the web page for an example.

Recall Sensitivity analysis in 2D. We have seen:

* Change in objective function coefficients (c_j).

  Q.1. for what range of values of c_j does the current optimal solution remain optimal?

  2. What if c_j changes beyond this range? → we'll address such questions about reoptimization now...
  Could we reoptimize quickly?

* Change in the right-hand side (rhs) coefficients (b_i).

  Q1. For what range of values of b_i does the current basis remain optimal?

  2. What if b_i changes beyond this range? Could we reoptimize quickly?

Now, we want to generalize to n-dimensions. We also want to consider

* change in constraint coefficients (A_{ij})
* Adding a column ...

In order to consider sensitivity analysis in higher dimensions, we first consider the simplex method in matrix form. This way, we will formalize many more properties of the tableau simplex method.

Simplex Method in Matrix Form

$$\left.\begin{array}{ll} \max & z=\bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}=\bar{b} \\ & \bar{x}\geq\bar{0} \end{array}\right\}$$ LP in standard form, after adding slack/excess/artificial vars.

A is $m\times n$, $\bar{b}$ is an m-vector, $\bar{c}$ and $\bar{x}$ are n-vectors.
$\bar{c}$ contains $-M$ terms for artificial variables.

$$\max\ z = \overbrace{\begin{bmatrix} — & \bar{c}^T & — \end{bmatrix}}^{n} \left.\begin{bmatrix} | \\ \bar{x} \\ | \end{bmatrix}\right\}n$$

$\bar{c}^T$: transpose of $\bar{c}$

s.t.
$$m\updownarrow \overbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}^{n} \begin{bmatrix} \\ \bar{x} \\ \\ \end{bmatrix} = \begin{bmatrix} | \\ \bar{b} \\ | \end{bmatrix}$$

$$\bar{x}\geq\bar{0}$$

Suppose we know which variables are basic in the optimal solution. Let $\bar{x}_B$ denote all the basic variables, and $\bar{x}_N$ the non-basic variables.

$$n\updownarrow \begin{bmatrix} \\ \bar{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \bar{x}_B \\ \hline \bar{x}_N \end{bmatrix} \begin{array}{l} \updownarrow m \\ \updownarrow n\text{-}m \end{array}$$

after possible reordering of the variables. Of course, there is no harm in reordering the x_j's, since order of addition is immaterial.

We split A and $\bar{c}$ in the same fashion into basic and non-basic parts.

$$\max \quad z = \overbrace{[\,\bar{c}_B^T \mid \bar{c}_N^T\,]}^{n} \left.\begin{bmatrix}\bar{x}_B\\ \bar{x}_N\end{bmatrix}\right\} n \qquad z = \bar{c}_B^T\bar{x}_B + \bar{c}_N^T\bar{x}_N$$

$$\text{s.t.} \qquad m\updownarrow \overbrace{\begin{bmatrix} B & \mid & N \end{bmatrix}}^{n} \begin{bmatrix}\bar{x}_B\\ \bar{x}_N\end{bmatrix} = \begin{bmatrix}\bar{b}\end{bmatrix} \qquad B\bar{x}_B + N\bar{x}_N = \bar{b}$$

$$\bar{x} \geqslant \bar{0}$$

Starting tableau T_0

$$\begin{array}{c|c|c|c|c} & z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline 1 & 1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\ \hline m & \bar{0} & B & N & \bar{b} \\ \hline & \multicolumn{4}{c}{\longleftarrow n+2 \longrightarrow} \end{array}$$

$$\xrightarrow[\text{or EROs}]{\text{simplex method}}$$

T^*: optimal tableau

$$\begin{array}{c|c|c|c} z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline 1 & 0 & ? & ? \\ \hline \bar{0} & I_m & ? & ? \end{array}$$

T_0: $(m+1) \times (n+2)$

Recall: How to invert an $m \times m$ matrix A:

$$[A \mid I_m] \xrightarrow{\text{EROs}} [I_m \mid A^{-1}]$$

We want to find the inverse of the $(m+1) \times (m+1)$ "basic" matrix

$$\left(\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}^{-1}\right) \underbrace{\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}}_{\text{"basic" matrix}} = \begin{bmatrix} 1 & 0 \\ \bar{0} & I_m \end{bmatrix}$$

The basis matrix B itself is invertible by definition, i.e., B^{-1} exists.

We claim $\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1} \end{bmatrix}$.

Let's check!

$$\begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix} = \begin{bmatrix} 1\times 1 + \bar{c}_B^T B \cdot \bar{0} & 1\times -\bar{c}_B^T + \bar{c}_B^T \overbrace{B^{-1} B}^{I_m} \\ \bar{0}\times 1 + B^{-1}\bar{0} & \bar{0}\cdot -\bar{c}_B^T + \underbrace{B^{-1}B}_{I_m} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \bar{0} & I_m \end{bmatrix}.$$

you should check the other multiplication, i.e., $\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}\begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1} \end{bmatrix} = I_{m+1}$, as well.

So, T^* is given by

$$\begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1} \end{bmatrix} T_0 = \begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1} \end{bmatrix} \begin{array}{c|c|c|c} z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline 1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\ \hline \bar{0} & B & N & \bar{b} \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|c} z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline 1 & 0 & -\bar{c}_N^T + \bar{c}_B^T B^{-1} N & \bar{c}_B^T B^{-1} \bar{b} \\ \hline \bar{0} & I_m & B^{-1} N & B^{-1}\bar{b} \\ \hline \end{array} = T^* \quad \text{(optimal tableau)}$$

Hence, $z^* = \bar{c}_B^T B^{-1} \bar{b}$, and the optimal solution is given by

$\bar{x}_B^* = B^{-1}\bar{b}$, i.e., $\bar{x}^* = \begin{bmatrix} \bar{x}_B^* \\ \bar{x}_N^* \end{bmatrix} = \begin{bmatrix} B^{-1}\bar{b} \\ \bar{0} \end{bmatrix} \begin{array}{l} \updownarrow m \\ \updownarrow n-m \end{array}$.

Hence, if we know which variables are going to be basic in the optimal tableau, we could convert the starting tableau to the optimal tableau through direct matrix multiplication.

Also, we can read off the elementary matrix from the optimal tableau!

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	0	$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N$	$\bar{c}_B^T B^{-1} \bar{b}$
$\bar{0}$	I	$B^{-1}N$	$B^{-1}\bar{b}$

If $N = I$, then the columns under $\bar{x}_N$ are

$$\begin{array}{c} \hline -\bar{c}_N^T + \bar{c}_B^T B^{-1} \\ \hline B^{-1} \\ \hline \end{array}$$

Further, if $\bar{c}_N = \bar{0}$, we have the elementary matrix!

In particular, we can read off B^{-1} from under the columns of slack and artificial variables in rows 1 to m in the optimal tableau!

We now look at an LP for which we write down the optimal tableau "directly" when the optimal basis is given. Our goal in studying the simplex method in this matrix form is to be able to do sensitivity analysis in a general form.

MATH 364 : Lecture 19 (10/22/2024)

Today: * sensitivity analysis in matrix form
- changing c_j when x_j is nonbasic
- changing c_j when x_j is basic

Recall: Simplex Method in matrix form:

starting tableau

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	$-\bar{c}_B^T$	$-\bar{c}_N^T$	0
$\bar{0}$	B	N	$\bar{b}$

$\xrightarrow{\text{EROs}}$

optimal tableau

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	0	$-\bar{c}_N^T+\bar{c}_B^T B^{-1} N$	$\bar{c}_B^T B^{-1} \bar{b}$
0	I_m	$B^{-1}N$	$B^{-1}\bar{b}$

$$\max\ z = -x_1 + x_2$$
$$\text{s.t.}\quad 2x_1 + x_2 \leq 4 \quad s_1$$
$$x_1 + x_2 \leq 2 \quad s_2$$
$$x_1, x_2 \geq 0$$

We are given $\{x_2, s_1\}$ in that order are optimal. Find the optimal tableau.

$BV = \{z, x_2, s_1\}$, $NBV = \{x_1, s_2\}$.

starting tableau

z	x_2	s_1	x_1	s_2	rhs
1	-1	$-\bar{c}_B^T$ 0	1	$-\bar{c}_N^T$ 0	0
0	1	1	2	0	4 $\bar{b}$
0	1	B 0	1 N	1	2

optimal tableau

z	x_2	s_1	x_1	s_2	rhs
1	0	0	$-\bar{c}_N^T+\bar{c}_B^T B^{-1}N$ 2	1	$\bar{c}_B^T B^{-1}\bar{b}$ 2
0	1	0	1 $B^{-1}N$	1	$B^{-1}\bar{b}$ 2
0	0	1	1	-1	2

$$\bar{c}^T = \begin{array}{cccc} x_2 & s_1 & x_1 & s_2 \\ \hline 1 & 0 & -1 & 0 \end{array}$$

$$-\bar{c}_B^T = \begin{array}{cc} x_2 & s_1 \\ \hline -1 & 0 \end{array}$$

$$-\bar{c}_N^T = \begin{array}{cc} x_1 & s_2 \\ \hline 1 & 0 \end{array}$$

$$B = \begin{array}{cc} x_2 & s_1 \\ \hline 1 & 1 \\ 1 & 0 \end{array} \Rightarrow B^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow B^{-1}N = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\bar{c}_B^T B^{-1} N = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N = \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

$$B^{-1}\bar{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and}$$

$$\bar{c}^T B^{-1}\bar{b} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2$$

We now consider sensitivity analysis using the matrix form of the simplex method. In preparation, we first write down the entries in the column of a variable x_j in the optimal tableau.

$$
\begin{array}{c|c|c|c}
z & \bar{x}_B & \bar{x}_N & \text{rhs} \\
\hline
1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\
\hline
\bar{0} & B & N & \bar{b} \\
\hline
\end{array}
\xrightarrow{\text{EROs}}
\overset{\text{optimal tableau}}{
\begin{array}{c|c|c|c}
z & \bar{x}_B & \bar{x}_N & \text{rhs} \\
\hline
1 & 0 & -\bar{c}_N^T + \bar{c}_B^T B^{-1} N & \bar{c}_B^T B^{-1} \bar{b} \\
\hline
0 & I_m & B^{-1}N & B^{-1}\bar{b} \\
\hline
\end{array}}
$$

Column of x_j in the optimal tableau:
where $\bar{a}_j$ is the column of x_j in A.

$$
\begin{array}{c}
x_j \\
\hline
-c_j + \bar{c}_B^T B^{-1} \bar{a}_j \\
\hline
B^{-1}\bar{a}_j \\
\hline
\end{array}
$$

This form applies for both non-basic and basic x_j's. If x_j is basic in Row-i, then $B^{-1}\bar{a}_j$ will be $\bar{e}_i$, the i^{th} m-unit vector. Also, $-c_j + \bar{c}_B^T B^{-1} \bar{a}_j = -c_j + c_j = 0$.

1. Changing c_j when x_j is non-basic

We change revenue/acre of wheat to \$25, so that x_2 is nonbasic at the optimal solution (and x_1 is indeed basic, which we will use in the next type of sensitivity analysis).

$$
\begin{array}{llll}
\max & z = 30x_1 + 25x_2 & & \text{(revenue)} \\
\text{s.t.} & x_1 + x_2 \le 7 & & \text{(land avail.)}\ s_1 \\
& 4x_1 + 10x_2 \le 40 & & \text{(labor hrs)}\ s_2 \\
& \cancel{10}x_1 \ge 3\cancel{0} & & \text{(min corn)}\ e_1 \\
& x_1, x_2 \ge 0 & & \text{(nonneg)}
\end{array}
$$

Can scale by 10 to get $x \ge 3$.

The optimal solution is at $A(7,0)$, with $z^* = 210$.

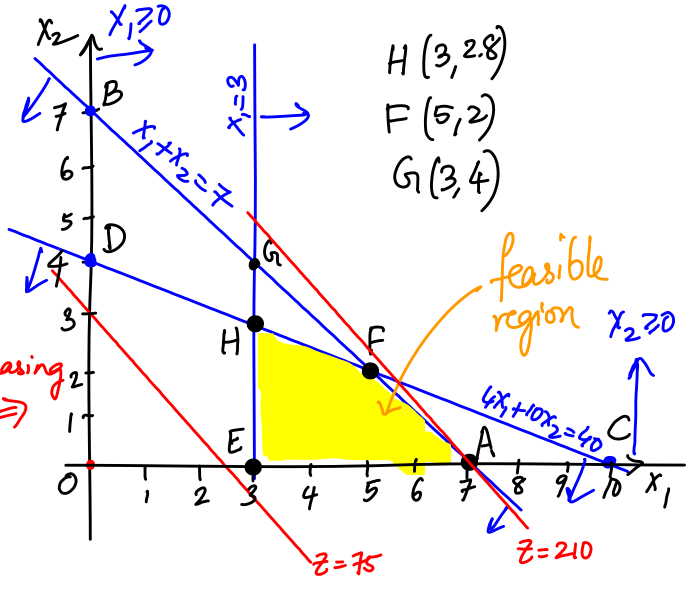


Tableau simplex:

BV	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	−30	−25	0	0	0	M	0
s_1	0	1	1	1	0	0	0	7
s_2	0	4	10	0	1	0	0	40
a_3	0	1	0	0	0	−1	1	3
	1	−M−30	−25	0	0	M	0	−3M
s_1	0	1	1	1	0	0	0	7
s_2	0	4	10	0	1	0	0	40
a_3	0	1	0	0	0	−1	1	3
	1	0	−25	0	0	−30	M+30	90
s_1	0	0	1	1	0	1	−1	4
s_2	0	0	10	0	1	4	−4	28
x_1	0	1	0	0	0	−1	1	3
	1	0	5	30	0	0	M	210
e_3	0	0	1	1	0	1	−1	4
s_2	0	0	6	−4	1	0	0	12
x_1	0	1	1	1	0	0	0	7

identity matrix under columns of s_1, s_2, a_3

$R_0 - MR_3$

$R_0 + (M+30)R_3$

B^{-1} is sitting in the columns that had I_3 in the starting tableau.

Optimal basis is $\{e_3, s_2, x_1\}$ in that order $(\equiv A(7,0))$.
So, x_2 is non-basic.

We can now write down the components of the optimal tableau as just described, i.e., $\bar{c}_B, \bar{c}_N, B^{-1}, B^{-1}\bar{b}, B^{-1}N$, etc.

$$\bar{c}_B^T = \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 30 \end{bmatrix}_{1\times 3} \end{array} \qquad B^{-1} = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3} \end{array}$$

→ how did we find B^{-1}?!

$$\Rightarrow \quad \bar{c}_B^T B^{-1} = [30 \quad 0 \quad 0].$$

We have I_3 (3x3 identity matrix) under the columns of s_1, s_2, a_3 in the starting tableau. And hence B^{-1} is sitting under these columns in the optimal tableau.

Recall: We have $B^{-1}N$ in the optimal tableau. Thus, if a submatrix of N is I (identity matrix), that submatrix will have B^{-1} in the optimal tableau. More generally, if a submatrix of A is I, then that submatrix is converted to B^{-1} in the optimal tableau.

Let's check to make sure B^{-1} is indeed correct. First, notice

$$B = \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \end{array},$$ the columns of e_3, s_2, x_1 from A, in that order.

Hence $$B^{-1}B = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ and

similarly, $$BB^{-1} = \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

You're welcome to use a package such as Octave (Matlab) or Python to do these matrix calculations. But you will not be tested on the use of such software package(s).

Suppose coefficient of x_2 in the objective function changes to $25+\Delta$.

Questions 1. For what range of values of Δ does the current basis remain optimal?

2. If for some Δ, the current basis is not optimal, how do we find the new optimal basis and solution (quickly)?

without starting from scratch, and resolving the LP all over again.

With $c_2 = 25+\Delta$, the entries in the x_2-column are

$$\begin{array}{c} x_2 \\ \hline -c_2+\bar{c}_B^T B^{-1}\bar{a}_2 \\ \hline B^{-1}\bar{a}_2 \\ \hline \end{array} \rightarrow \begin{array}{c} x_2 \\ \hline -(25+\Delta) + \begin{bmatrix}30 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\10\\0\end{bmatrix} \\ \hline \begin{bmatrix}1 & 0 & -1\\ -4 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\10\\0\end{bmatrix} \\ \hline \end{array} \rightarrow \begin{array}{c} x_2 \\ \hline 5-\Delta \\ \hline 1 \\ 6 \\ 1 \\ \hline \end{array}$$

Current basis remains optimal as long as $5-\Delta \geq 0$, i.e., $\Delta \leq \boxed{5}$

"reduced cost" of wheat

The current solution remains optimal as well for $\Delta \leq 5$.

Def The **reduced cost** of a non-basic variable (in a max-LP) is the maximum amount by which its objective function coefficient can be increased with the current basis remaining optimal.

If the objective function coefficient of a nonbasic variable increases by more than its reduced cost, the variable can enter the basis, and improve the value of z. At this point, the current basis becomes suboptimal. Here, we could pivot this non-basic variable into the basis from the current optimal tableau (and not start from scratch again).

Consider $\Delta = 7$ here, for instance. We could pivot x_2 into the basis, and obtain the new optimal tableau in one (new) pivot.

$5-\Delta$

	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	0	-2	30	0	0	M	210
e_3	0	0	1	1	0	1	-1	4
s_2	0	0	6	-4	1	0	0	12
x_1	0	1	1	1	0	0	0	7
	1	0	0	$86/3$	$1/3$	0	M	214
e_3	0	0	0	$5/3$	$-1/6$	1	-1	2
x_2	0	0	1	$-2/3$	$1/6$	0	0	2
x_1	0	1	0	$5/3$	$-1/6$	0	0	5

New $z^* = 214$ (at $F(5,2)$).

Notice that once the revenue/acre of wheat is \$32, which is higher than the revenue/acre of corn (still at \$30), it makes sense to farm both wheat and corn.

2. Changing c_j when x_j is basic

Consider changing c_1 (coefficient of x_1) from 30 to $30+\Delta$. Since an entry in $\bar{C}_B^T$ is changing here, more entries in Row-0 under the non-basic columns could change as compared to the case when we were changing a non-basic c_j.

Now, $\bar{C}_B^T = \left[\begin{array}{ccc} e_3 & s_2 & x_1 \\ 0 & 0 & 30+\Delta \end{array}\right]_{1\times 3}$ $B^{-1} = \left[\begin{array}{ccc} s_1 & s_2 & \\ 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]_{3\times 3}$, and hence

$$\bar{C}_B^T B^{-1} = [30+\Delta \quad 0 \quad 0].$$

Current basis remains optimal as long as $c_j' \geq 0$ for all j (i.e., the numbers in Row-0 remain ≥ 0).

$c_j' = 0$ if x_j is basic, and hence we concentrate on the non-basic entries.

For the non-basic variables,

$$-\bar{C}_N^T + \bar{C}_B^T B^{-1} N = \left[\begin{array}{ccc} x_2 & s_1 & a_3 \\ -25 & 0 & M \end{array}\right] + [30+\Delta \quad 0 \quad 0]\left[\begin{array}{ccc} x_2 & s_1 & a_3 \\ 1 & 1 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$= [-25 \quad 0 \quad M] + [30+\Delta \quad 30+\Delta \quad 0]$$

$$= [5+\Delta \quad 30+\Delta \quad M]$$

$\Rightarrow$ Current basis remains optimal as long as

$$[5+\Delta \quad 30+\Delta \quad M] \geq \bar{0}^T$$

$$\Rightarrow \quad \left.\begin{array}{ll} 5+\Delta \geq 0 & \Rightarrow \Delta \geq -5 \\ 30+\Delta \geq 0 & \Rightarrow \Delta \geq -30 \end{array}\right\} \Rightarrow \boxed{\Delta \geq -5.}$$

As long as revenue per acre of corn is at least \$25, which is the same as that for wheat, we continue to farm corn in all 7 acres.

If $\Delta = -8$, for instance, we can find the <u>updated tableau</u> (non.optimal) for that value of Δ, and continue the simplex method from there.

MATH 364: Lecture 20 (10/24/2024)

Today:
- * change of b_i
- * shadow price
- * changing column of x_j

Recall: simplex method in matrix form:

$$\begin{array}{c|c|c|c}
z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline
1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\ \hline
\bar{0} & B & N & \bar{b}
\end{array}
\xrightarrow{\text{EROs}}
\begin{array}{c|c|c|c}
z & \bar{x}_B & \bar{x}_N \ \geq 0 & \text{rhs} \\ \hline
1 & 0 & \boxed{-\bar{c}_N^T+\bar{c}_B^T B^{-1}N} & \bar{c}_B^T B^{-1}\bar{b} \\ \hline
0 & I_m & B^{-1}N & \boxed{B^{-1}\bar{b}} \geq 0
\end{array}$$

The current basis remains optimal as long as

1. $-\bar{c}_N^T+\bar{c}_B^T B^{-1}N \geq \bar{0}^T$ and (optimality for max-LP)
2. $B^{-1}\bar{b} \geq \bar{0}$. (feasibility)

3. Changing the right-hand side (rhs) of a constraint (b_i)

$b_i \longrightarrow b_i+\Delta$, so only rhs column is changed.

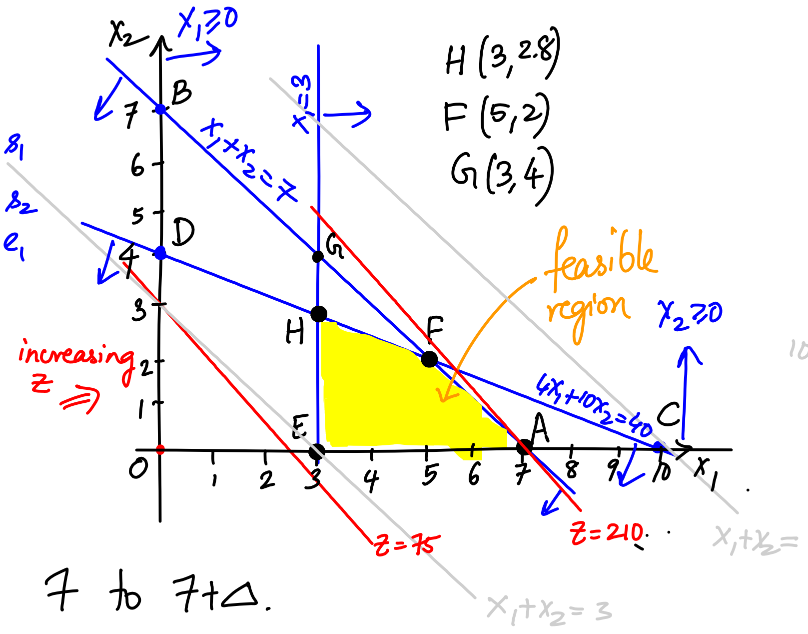
The current basis remains optimal as long as $B^{-1}\bar{b} \geq \bar{0}$ (feasibility).

Note that Row-0 numbers are not affected.

It is helpful to recall the graphical solution:

$$\begin{array}{llll}
\max & z = 30x_1+25x_2 & & \text{(revenue)} \\
\text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land avail.)} \ s_1 \\
 & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \ s_2 \\
 & \cancel{1}0x_1 & \geq 3\cancel{0} & \text{(min corn)} \ e_1 \\
 & x_1, x_2 & \geq 0 & \text{(nonneg)}
\end{array}$$

Can scale by 10 to get $x_1 \geq 3$.



The optimal solution is

$$z^* = 210.$$

Consider changing # acres from 7 to $7+\Delta$.

$$\bar b = \begin{bmatrix}7\\40\\3\end{bmatrix} \longrightarrow \begin{bmatrix}7+\Delta\\40\\3\end{bmatrix} = \begin{bmatrix}7\\40\\3\end{bmatrix} + \Delta \underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix}}_{\bar e_1 \text{, the first unit vector}}$$

More generally, when we change $b_i \to b_i + \Delta$, the new rhs vector is $\bar b =$ old $\bar b + \Delta \bar e_i$, where $\bar e_i$ is the i^{th} unit vector.

$$\bar e_i = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} \rightarrow i$$

New rhs in optimal tableau is given by

$$B^{-1}\underbrace{(\bar b + \Delta \bar e_i)}_{\text{new } \bar b} = \left[\begin{array}{ccc} s_1 & s_2 & a_3 \\ 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \left(\begin{bmatrix}7\\40\\3\end{bmatrix} + \Delta \underset{\bar e_1}{\begin{bmatrix}1\\0\\0\end{bmatrix}} \right) = \underset{\substack{\downarrow \\ \text{original} \\ \text{optimal } \bar x_B}}{B^{-1}\bar b} + \Delta \underset{\substack{\downarrow \\ 1^{st} \text{ column} \\ \text{of } B^{-1}}}{\begin{bmatrix}1\\-4\\1\end{bmatrix}}$$

$$= \begin{bmatrix}4\\12\\7\end{bmatrix} + \Delta \begin{bmatrix}1\\-4\\1\end{bmatrix} = \begin{bmatrix}4+\Delta\\12-4\Delta\\7+\Delta\end{bmatrix} \longrightarrow \text{new } \bar x_B$$

More generally, new $B^{-1}\bar b =$ old $B^{-1}\bar b + \Delta B^{-1} \bar e_i$

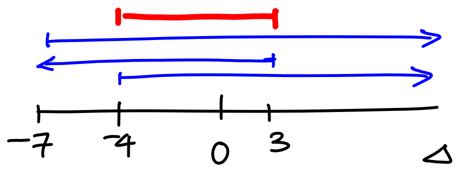
$$= \text{old } B^{-1}\bar b + \Delta \underbrace{[B^{-1}]_{\cdot i}}_{i^{th} \text{ column of } B^{-1}}$$

We need the new $\bar x_B = \begin{bmatrix}4+\Delta\\12-4\Delta\\7+\Delta\end{bmatrix} \geq \bar 0$ for feasibility, and hence optimality.

$\Rightarrow$ $4+\Delta \geq 0$, $12-4\Delta \geq 0$, and $7+\Delta \geq 0$

$$\Rightarrow \Delta \geq -4, \quad \Delta \leq 3, \quad \text{and} \quad \Delta \geq -7.$$

$$\Longrightarrow \boxed{-4 \leq \Delta \leq 3.}$$



As long as there are at least 3 acres ($\Delta=-4$), and at most 10 acres ($\Delta=3$), we will continue to farm only corn in all of the land. It so happens that in this case, even when $b_1=11$, say, i.e., $\Delta=4$, we would still farm only corn. But the (land available) constraint will no longer be binding, as we have enough labor hours to farm corn in at most 10 acres.

Shadow price of (land) constraint:

$$\text{New objective function value} = \bar{C}_B^T B^{-1}(\text{new } \bar{b}) = \bar{C}_B^T(\underbrace{B^{-1}\cdot \text{new } \bar{b}}_{\text{new } \bar{x}_B})$$

$$= \begin{array}{ccc} e_3 & s_2 & x_1 \\ \left[0\right. & 0 & \left.30\right] \end{array} \left(\begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \Delta \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right)$$

$$\bar{C}_B^T = \begin{array}{ccc} e_3 & s_2 & x_1 \\ \left[0\right. & 0 & \left.30\right] \end{array}$$

$$= 210 + \underset{\text{shadow price}}{\textcircled{30}}\,\Delta$$

The shadow price of land constraint = \$30.

Jones would pay upto \$30 for one extra acre of land.

Notice that this price is equal to the revenue from an acre of corn.

If Δ is outside this range, the $\bar{x}_B$ is no longer feasible. The rhs will no longer be ≥ 0, but you can use a dual simplex pivot to reoptimize quickly. More on this topic after we introduce linear programming duality.

Now let's change (# labor hrs) from 40 to 40+Δ. Thus, we are changing $b_2 \rightarrow b_2+\Delta$, and hence

$$\text{new } \bar{b} = \text{old } \bar{b} + \Delta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \bar{e}_2 \text{ (2nd unit vector)}$$

$$\Rightarrow \text{New } \bar{x}_B = \text{old } \bar{x}_B + \Delta (2^{nd} \text{ column of } B^{-1})$$

$$= \begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \Delta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 12+\Delta \\ 7 \end{bmatrix} \geq \bar{0} \text{ for feasibility.}$$

$$\Rightarrow 12+\Delta \geq 0 \Rightarrow \boxed{\Delta \geq -12.}$$

Shadow price:

$$\text{New } z^* = \bar{c}_B^T (\text{new } \bar{x}_B) = [0 \ 0 \ 30] \left(\begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \Delta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 210 + (0)\Delta.$$

$\Rightarrow$ Shadow price is zero here, as we are not using all of the 40 hours of labor available (we're using only 28 hrs of labor).

If $\Delta < -12$ here, the current solution becomes infeasible. To get the new optimal solution, we need to do a dual simplex pivot.

(more on this method after we introduce LP duality)

4. Changing the column of a nonbasic variable x_j

Consider changing the revenue/acre of wheat (x_2) from 25 to 35, and at the same time changing the # labor hrs/acre of wheat from 10 to 8. Recall x_2 is non-basic in the optimal tableau. Here is how the column of x_2 in the starting tableau changes:

$$\begin{array}{c} x_2 \\ \hline -25 \\ \hline 1 \\ 10 \\ 0 \\ \hline \end{array} \longrightarrow \begin{array}{c} x_2 \\ \hline -35 \\ \hline 1 \\ 8 \\ 0 \\ \hline \end{array}$$

We can find the column of x_2 in the modified/new optimal tableau directly (using $\bar{c}_B$, B^{-1} from the optimal tableau).

Recall, $\bar{c}_B^T B^{-1} = [30 \quad 0 \quad 0]$ still.

updated column of x_2 in optimal tableau

$$\begin{array}{c} x_j \\ \hline -c_j + \bar{c}_B^T B^{-1} \bar{a}_j \\ \hline B^{-1} \bar{a}_j \\ \hline \end{array} \qquad \begin{array}{c} x_2 \\ \hline -35 + [30 \quad 0 \quad 0] \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} \\ \hline \end{array} \qquad \begin{array}{c} x_2 \\ \hline -5 \\ \hline 1 \\ 4 \\ 1 \\ \hline \end{array}$$

new $\bar{a}_2$

Since coefficient of x_2 is Row-0 is not ≥ 0, new tableau is not optimal. But we can reoptimize quickly:

updated column of x_2

	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	0	-5	30	0	0	M	210
e_3	0	0	1	1	0	1	-1	4
s_2	0	0	4	-4	1	0	0	12
x_1	0	1	1	1	0	0	0	7
	1	0	0	25	$5/4$	0	M	225
e_3	0	0	0	2	$-1/4$	1	-1	1
x_2	0	0	1	-1	$1/4$	0	0	3
x_1	0	1	0	2	$-1/4$	0	0	4

New optimal solution is $x_1=4, x_2=3, z^*=225$.

A similar approach can be used when considering a new variable. For instance, Jones could consider growing barley that gives a revenue of \$35/acre and uses 8 hrs/acre of labor. Should he grow any barley? The answer is Yes.

$$\begin{array}{c} x_3 \\ \hline -c_3+\bar{c}_B^T B^{-1}\bar{a}_3 \\ \hline B^{-1}\bar{a}_3 \\ \hline \end{array} \longrightarrow \begin{array}{c} x_3 \\ \hline -5 \\ \hline 1 \\ 4 \\ 1 \\ \hline \end{array}$$

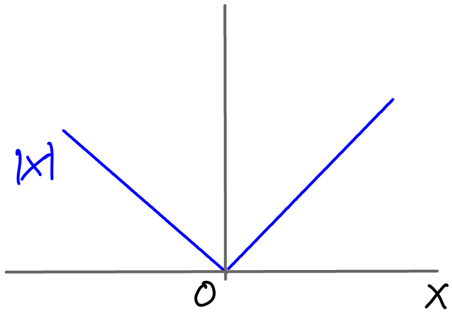
using the same calculations done in the previous page.

So we can add this column of x_3 into the original optimal tableau and pivot it in to find the new optimal tableau.

MATH 364: Lecture 21 (10/29/2024)

Today: * LP duality
* motivation for dual LP

Homework 8 Problems



1. The objective function is not linear. $f(x)=|x|$ is a piecewise linear function.

$$f(x)=|x|=\begin{cases} x & \text{if } x\geqslant 0, \\ -x & \text{if } x<0 \end{cases}$$

Try using idea for modeling urs variables $x \rightarrow x^+, x^-$

Or, try solving two LPs with two (related) objective functions.

2. Consider columns of x_i^+, x_i^- $\qquad x_i \leftarrow x_i^+ - x_i^-$

x_i^+	x_i^-
c_i	$-c_i$
a_1	$-a_1$
a_2	$-a_2$
$\vdots$	$\vdots$
a_m	a_m

Show what happen under scaling and replacement EROs $\left(\frac{1}{a_j}\right)R_j$ and $R_k + \alpha R_j$.

Show that the opposite sign property is maintained under both such EROs.

3.

	x_ℓ	x_e
	$\textcircled{0}$	c
	0	a_1
	$\vdots$	a_2
i	1	α
	0	$\vdots$
	0	a_m

what happens after pivot?

Let x_ℓ leave and x_e enter in its place.

* $\alpha>0$, so that x_e can be pivoted into Row-i.

* $c\leq 0$ to start with.

LP Duality

Associated with every LP (linear program) is another LP called its **dual LP**. The original LP is called the **primal LP**. There are important relationships between the primal and dual LPs, both from the mathematical as well as economic points of view.

A max LP in which every constraint is $\leq$ and all variables are ≥ 0 is a **normal max LP**. Similarly, a min LP in which every constraint is $\geq$ and all variables are ≥ 0 is a **normal min LP**.

A $\geq$ constraint is hence normal for a min-LP, but is opposite to normal for a max LP. Similarly, a $\leq$ constraint is normal for a max-LP, and is opposite to normal for a min-LP.

Nonnegative variables (≥ 0) are always normal (for both max- and min-LPs).

Intuition for normal LPs

- max revenue s.t. upper bounds on raw materials, i.e., $\leq$ constraints
- min cost s.t. demands (min. requirements), i.e., $\geq$ constraints

Nonnegative variables are always normal.

Let's write the dual LP of a normal max-LP. This dual LP will be a normal min-LP.

Example : Find the dual of the following LP:

(P)
for primal

$$
\begin{array}{lll}
\overset{\min}{\max} & z = 2x_1 + x_2 & \\
\text{s.t.} & -x_1 + x_2 \leq 1 & y_1 \\
 & x_1 + x_2 \leq 3 & y_2 \\
 & x_1 - 2x_2 \leq 4 & y_3 \\
 & x_1 , x_2 \geq 0 &
\end{array}
$$

(D)
for dual

$$
\begin{array}{ll}
\min & 1y_1 + 3y_2 + 4y_3 \\
\text{s.t.} & -y_1 + y_2 + y_3 \geq 2 \\
 & y_1 + y_2 - 2y_3 \geq 1 \\
 & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0
\end{array}
$$

Primal (P)	⟷	Dual (D)
constraint i $(\overset{\text{normal}}{\leq})$	⟷	variable $y_i \geq 0$ → normal
variable x_j $(\overset{\text{normal}}{\geq 0})$	⟷	constraint j $(\geq)^{\text{normal}}$
objective : max	⟷	objective : min

Dual of a general form normal max-LP

(P)

$$
\begin{array}{lll}
\max & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 & y_1 \\
 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 & y_2 \\
 & \vdots \qquad\qquad\qquad\qquad \vdots & \vdots \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m & y_m \\
 & x_1 ,\ x_2 , \ldots ,\ x_n \geq 0 &
\end{array}
$$

(D)

$$
\begin{array}{ll}
\min & w = b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{s.t.} & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
 & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2 \\
 & \vdots \\
 & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n \\
 & y_1 ,\ y_2 , \ldots ,\ y_m \geq 0
\end{array}
$$

"Dual of a deal is Primal" : if you take the dual of the dual LP of a given LP, you get the original LP back.

Primal-Dual Relationships

	min LP		max LP	
variables	≥ 0 normal	⟷	$\leq$ normal	constraints
	≤ 0 opposite to normal	⟷	$\geq$ opposite to normal	
	urs	⟷	$=$	
constraints	$\geq$ normal	⟷	≥ 0 normal	variables
	$\leq$ opposite to normal	⟷	≤ 0 opposite to normal	
	$=$	⟷	urs	

In general,

normal variables in (P) ⟷ normal constraints in (D)

Opposite to normal variables in (P) ⟷ opposite to normal constraints in (D)

urs variables in (P) ⟷ = constraints in (D)

You should **not** try to memorize the above table of primal-dual relationships. Instead, use the idea of normal variables/constraints corresponding to normal constraints.

To stress this point, we will rewrite this table in other equivalent forms.

Write the dual LP for the given LPs

1. (P)

$$\begin{aligned} \min\ & z = x_1 - x_2 \\ \text{s.t.}\ & 2x_1 + x_2 \geq 4 \quad y_1 \geq 0 \\ & x_1 + x_2 \geq 1 \quad y_2 \geq 0 \\ & x_1 + 2x_2 \leq 3 \quad y_3 \leq 0 \\ & x_1,\ x_2 \geq 0 \\ & \leq \quad \leq \end{aligned}$$

(D)

$$\begin{aligned} \max\ & w = 4y_1 + y_2 + 3y_3 \\ \text{s.t.}\ & 2y_1 + y_2 + y_3 \leq 1 \\ & y_1 + y_2 + 2y_3 \leq -1 \\ & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \leq 0 \end{aligned}$$

2. (P)

$$\begin{aligned} \min\ & w = 4y_1 + 2y_2 - y_3 \\ \text{s.t.}\ & y_1 + 2y_2 \leq 6 \quad y_1 \leq 0 \\ & y_1 - y_2 + 2y_3 = 8 \quad y_2 \text{ urs} \\ & y_1,\ y_2 \geq 0,\ y_3 \text{ urs} \\ & \leq \quad \leq \quad = \end{aligned}$$

could use $\{x_1, x_2\}$ or $\{u_1, u_2\}$ ...

(D)

$$\begin{aligned} \max\ & w = 6y_1 + 8y_2 \\ \text{s.t.}\ & y_1 + y_2 \leq 4 \\ & 2y_1 - y_2 \leq 2 \\ & 2y_2 = -1 \\ & y_1 \leq 0,\ y_2 \text{ urs} \end{aligned}$$

3. (P)

$$\begin{aligned} \max\ & z = 3x_2 - 4x_1 + 2x_3 \\ \text{s.t.}\ & 2x_1 + 0.5x_3 + 7x_2 \geq 5 \quad y_1 \leq 0 \\ & -3x_2 + 5x_1 \leq -3 \quad y_2 \geq 0 \\ & 2x_1 + 6x_3 = 2 \quad y_3 \text{ urs} \\ & x_4 \geq 5 \quad y_4 \leq 0 \\ & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs},\ x_4 \geq 0 \\ & \geq \quad \leq \quad = \quad \geq \end{aligned}$$

(D)

$$\begin{aligned} \min\ & w = 5y_1 - 3y_2 + 2y_3 + 5y_4 \\ \text{s.t.}\ & 2y_1 + 5y_2 + 2y_3 \geq -4 \\ & 7y_1 - 3y_2 \leq 3 \\ & 0.5y_1 + 6y_3 = 2 \\ & y_4 \geq 0 \\ & y_1 \leq 0,\ y_2 \geq 0,\ y_3 \text{ urs},\ y_4 \leq 0 \end{aligned}$$

Notice the variables might not be ordered (or arranged) nicely. But you just have to read down the column of each variable to get the corresponding constraint in the dual.

Motivation behind the dual (how and why)

Farmer Jones LP

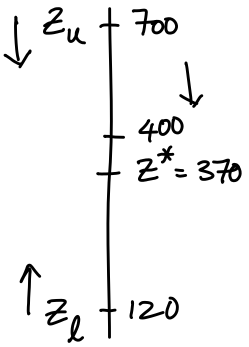
max $z = 30x_1 + 100x_2$ (total revenue)

s.t. $x_1 + x_2 \leq 7$ (land)

$4x_1 + 10x_2 \leq 40$ (labor hrs)

ignore for now, just for interpretation ~~$10x_1 \geq 30$ (min corn)~~

$x_1, x_2 \geq 0$ (non-neg)



Optimal solution: $x_1 = 3, x_2 = 2.8, z^* = 370$

Let z_u = upper bound on z^*, z_ℓ = lower bound on z^*.

This is a standard approach to many optimization problems - start with lower and upper bounds for the quantity you are optimizing, and tighten these bounds.

Any feasible point (x_1, x_2) gives a lower bound, e.g., $x_1 = 4, x_2 = 0$, giving $z_\ell = 120$.

→ we could consider (7,0) or (4.5,1.5) or (5,2), or ...

How do we get (an) upper bound?

Consider $100 \times$ (land): $100x_1 + 100x_2 \leq 700$. But

$$z = 30x_1 + 100x_2 \leq 100x_1 + 100x_2 \leq 700$$

as long as $x_1, x_2 \geq 0$ (which is true here).

Notice also that the coefficients of x_1, x_2 in z should compare in the right way with the coefficients in 100(land).

The goal is to get smaller and smaller z_u values. May be the (Labor hrs) constraint could give us a smaller z_u value.

10 x (Labor hrs): $40x_1 + 100x_2 \le 400$

Again $z = 30x_1 + 100x_2 \le 40x_1 + 100x_2 \le 400 \longrightarrow$ new z_u

Let's multiply the (land) and (labor hrs) constraints by y_1 and y_2. We need $y_1 \geqslant 0$, $y_2 \geqslant 0$, as the sense of the scaled inequality should stay as $\le$. We want to say

$$z \le 7y_1 + 40y_2 .$$

But we must be able to compare the coefficients of x_1 and x_2 to those in z properly:

$$\begin{array}{lll} y_1(x_1 + x_2 \le 7) & \text{(land)} & + \\ y_2(4x_1 + 10x_2 \le 40) & \text{(labor hrs)} & \\ \hline \end{array}$$

$$\begin{array}{ccccccl} & (y_1+4y_2) & x_1 & + & (y_1+10y_2) & x_2 & \le 7y_1 + 40y_2 = w \\ & \text{IV} & & & \text{IV} & & \\ z = & 30 & x_1 & + & 100 & x_2 & \end{array}$$

So we need $y_1 + 4y_2 \geqslant 30$ and $y_1 + 10y_2 \geqslant 100$. Also, we want to find the smallest upper bound $w = 7y_1 + 40y_2$. Combining all these requirements gives the dual LP!

$$\begin{array}{lll} \min & w = 7y_1 + 40y_2 & \\ \text{s.t.} & y_1 + 4y_2 \geqslant 30 & \\ & y_1 + 10y_2 \geqslant 100 & (D) \\ & y_1,\ y_2 \geqslant 0 & \end{array}$$

MATH 364: Lecture 22 (10/31/2024)

Today: * economic interpretation of dual LP
* Duality in matrix form - results

Economic Interpretation of the dual LP for a max LP

Consider the (original) Farmer Jones LP:

$$
\begin{array}{llll}
\max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\
\text{s.t.} & x_1 + x_2 \leq 7 & y_1 \geq 0 & \text{(land)} \\
 & 4x_1 + 10x_2 \leq 40 & y_2 \geq 0 & \text{(labor hrs)} \\
 & \cancel{10x_1 \geq 30} & \cancel{y_3 \leq 0} & \cancel{\text{(min corn)}} \\
 & x_1, x_2 \geq 0 & & \text{(non-neg)} \\
 & \geq \quad \geq & &
\end{array}
$$

ignore for now, just for interpretation

$$
\begin{array}{ll}
\min & w = 7y_1 + 40y_2 + 30y_3 \\
\text{s.t.} & y_1 + 4y_2 + 10y_3 \geq 30 \\
 & y_1 + 10y_2 \geq 100 \\
 & y_1 \geq 0, y_2 \geq 0, y_3 \leq 0
\end{array}
$$

We will deal with the (min.corn) constraint, which is opposite to normal, after we explain the rest of the problem.

Suppose a firm wants to buy the farming enterprise from Jones. The firm needs to make an offer, i.e., unit price, for every acre and every labor hour Jones has. The firm would like to buy Jones' enterprise at minimum cost. Thus, the firm quotes prices y_1 and y_2 for each acre of land and hour of labor, respectively. The total cost for the firm is hence $w = 7y_1 + 40y_2$, and it tries to minimize w.

Hence its objective function is

$$\min \quad w = 7y_1 + 40y_2 \qquad \text{(cost)}$$

At the same time, the offer should be attractive to Jones.

If Jones has 1 acre of land and 4 hrs of labor, he can farm corn in that acre and make \$30 revenue.
Hence the prices the firm offers should be such that they match this revenue, i.e.,

$$y_1 + 4y_2 \geq 30 \qquad \text{(match revenue from corn)}$$

Similarly, for wheat, we should have

$$y_1 + 10y_2 \geq 100 \qquad \text{(match revenue from wheat)}$$

y_1, y_2 are unit prices quoted by the firm, so should be non negative.

Putting it all together, we get the dual LP, which captures the problem from the competing firm's perspective.

$$\begin{aligned} \min \quad & w = 7y_1 + 40y_2 && \text{(total cost)} \\ \text{s.t.} \quad & y_1 + 4y_2 \geq 30 && \text{(match revenue from corn)} \\ & y_1 + 10y_2 \geq 100 && \text{(match revenue from wheat} \\ & y_1, y_2 \geq 0 && \text{(non-neg)} \end{aligned}$$

What about (min-corn) constraint?

We first modify the dual LP to include y_3:

$$\begin{aligned} \min \quad & w = 7y_1 + 40y_2 + 30y_3 \\ \text{s.t.} \quad & y_1 + 4y_2 + 10y_3 \geq 30 \\ & y_1 + 10y_2 \geq 100 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \leq 0 \end{aligned}$$

(D) with y_3 included

Jones was making 30 bushels of corn/week. Hence the firm could sell off those 30 (or more) bushels of corn at a price of $-y_3$, and make some revenue that offsets its total cost.

$\rightarrow -y_3$ because it is in the reverse sense of y_1 and y_2

Hence, it will sell each bushel of corn at $-y_3$ dollars, such that $y_3 \leq 0$.

Economic Interpretation of the dual of a (normal) min-LP

Gaseous Chemicals LP (from HW2)

3. (25) Gaseous Chemicals makes three chemicals A, B, and C, via two processes. Running Process 1 for an hour costs \$4, and yields 3, 1, and 1 units of A, B, and C, respectively. Running Process 2 for an hour costs \$1, and yields 1 unit of A and 1 unit of B. At least 10, 5, and 3 units of A, B, and C, respectively, must be produced in order to meet demand. Determine the daily production plan that minimizes the total daily cost for meeting the demands of Gaseous Chemicals using the graphical method to solve LPs.

(P)

$$
\begin{array}{llcrl}
\min\ z = & 4x_1 + x_2 & & & \text{(total cost)} \\
\text{s.t.} & 3x_1 + x_2 & \ge & 10 & \text{(demand A)}\ y_1 \geqslant 0 \\
& x_1 + x_2 & \ge & 5 & \text{(demand B)}\ y_2 \geqslant 0 \\
& x_1 & \ge & 3 & \text{(demand C)}\ y_3 \geqslant 0 \\
& x_1,\ x_2 & \ge & 0 & \text{(non-negativity)} \\
& \le\ \ \le & & &
\end{array}
$$

(D)

$$
\begin{array}{ll}
\max & w = 10y_1 + 5y_2 + 3y_3 \\
\text{s.t.} & 3y_1 + y_2 + y_3 \le 4 \\
& y_1 + y_2 \le 1 \\
& y_1,\ y_2,\ y_3 \geqslant 0
\end{array}
$$

Suppose a firm wants to sell chemicals A, B, C to Gaseous. The firm quotes unit prices y_1, y_2, y_3 for A, B, C. The firm tries to maximize its revenue: $W = 10y_1 + 5y_2 + 3y_3$. Gaseous would not buy more than 10 units of A, which is the demand for A, and similarly for B and C.

The idea here is that Gaseous could buy the finished products (chemicals A, B, C) from the other firm to meet the corresponding demands, rather than make the products themselves by running Processes 1 and 2.

The offer should be attractive to Gaseous. If Gaseous has \$4, they could run Process 1 for 1 hour and make 3 units of A, 1 unit of B, and 1 unit of C. Hence the total amount Gaseous has to pay for 3,1,1 units of A,B,C, respectively cannot be more than \$4. Hence

$3y_1 + y_2 + y_3 \leq 4$ (match Process 1 cost)

Similarly, for Process 2, we get $y_1 + y_2 \leq 1$
(match Pr2 cost)

y_1, y_2, y_3 are unit prices, so should be ≥ 0.

If there is an opposite-to-normal constraint, i.e., a $\leq$ constraint, in the primal min-LP, you could interpret it in a way similar to how we interpreted the (min-corn) constraint in the Farmer Jones LP. The second firm would have to pay for this raw material here, for instance, and would quote a unit price for the same, which would be ≤ 0 as compared to y_1, y_2, y_3 here.

Duality in Matrix Form

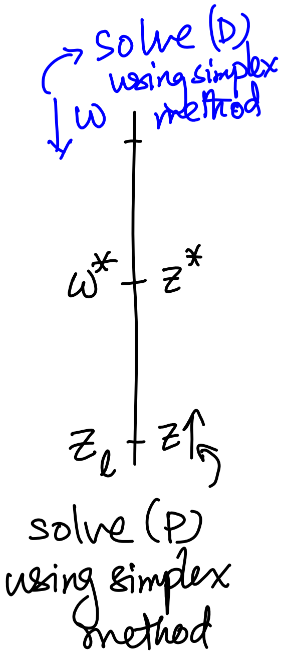
→ to explore the connections between the primal and dual LPs in depth, the primal and dual LPs in depth, we switch to matrix notation now.

(P) $\max\ z = \bar{c}^T\bar{x}$
s.t. $A\bar{x} \leq \bar{b}$ $\quad \bar{y} \geq \bar{0}$ → m-vector
$\bar{x} \geq \bar{0}$
$\geq$

normal max-LP

$\min\ w = \bar{b}^T\bar{y}$
$A^T\bar{y} \geq \bar{c}$ (D)
$\bar{y} \geq \bar{0}$

normal min-LP



weak because it specifies a '$\leq$' result as opposed to an '$=$' result

Lemma 1 (weak duality) If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), we have $z = \bar{c}^T\bar{x} \leq \bar{b}^T\bar{y} = w$.

Proof $\bar{x}$ is feasible for (P) $\Rightarrow$ $A\bar{x} \leq \bar{b}$, $\bar{x} \geq \bar{0}$

$\bar{y}$ is feasible for (D) $\Rightarrow$ $A^T\bar{y} \geq \bar{c}$, $\bar{y} \geq \bar{0}$

$$\bar{y}^T(A\bar{x} \leq \bar{b}) \Rightarrow \quad \bar{y}^T A\bar{x} \leq \bar{y}^T\bar{b} = \bar{b}^T\bar{y} = w.$$

$$(A^T\bar{y} \geq \bar{c})^T \Rightarrow \quad (\bar{y}^T A \geq \bar{c}^T)\bar{x} \Rightarrow \bar{y}^T A\bar{x} \geq \bar{c}^T\bar{x} = z.$$

Combining, we get $z = \bar{c}^T\bar{x} \leq \bar{y}^T A\bar{x} \leq \bar{b}^T\bar{y} = w$. □

denotes "end of proof"
also "Q.E.D."

The result (of weak duality) holds for general primal-dual LP pairs, not just for normal LPs.

Exploiting the primal-dual relationships is a standard practice in solving most optimization problems. Hence, we could solve (P) to optimality, or, alternatively, we could solve (D) to optimality. As the next Lemma states, solving one of them to optimality is guaranteed to equivalently solve the other problem as well.

But another versatile idea is to combine the solution process for both (P) and (D). Thus, one could switch back and forth between (P) and (D), and tighten both bounds simultaneously. Such methods are called primal-dual algorithms.

Lemma 2 (Strong duality) If $z = \bar{c}^T\bar{x} = \bar{b}^T\bar{y} = w$ for $\bar{x}, \bar{y}$ feasible for (P) and (D), respectively, then $\bar{x}, \bar{y}$ are optimal for (P) and (D), respectively.

Proof All z values lie below all w values (Lemma 1). Hence when $z=w$, we get optimality for both.

MATH 364: Lecture 23 (11/05/2024)

Today:
- * dual theorem
- * reading off optimal $\bar{y}$ from primal tableau
- * shadow $\text{price}_i := y_i$

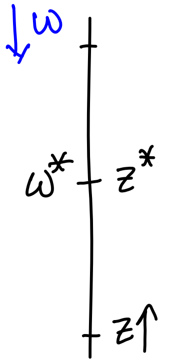
We first present two more results that connect unboundedness of an LP with the infeasibility of its dual LP.

Recall: Lemma 1 (weak duality) $z = \bar{c}^T\bar{x} \leq \bar{b}^T\bar{y} = w$ for $\bar{x}\in(P), \bar{y}\in(D)$

Lemma 2 (strong duality) If $z = \bar{c}^T\bar{x} = \bar{b}^T\bar{y} = w$ then $\bar{x}, \bar{y}$ are optimal for (P) and (D), respectively.

Lemmas 3 and 4 If (P) is unbounded, then (D) is infeasible. Similarly, if (D) is unbounded, then (P) is infeasible.

If (P) is unbounded, we can push z up without limits. Hence there are no finite w values, i.e., there are no feasible $\bar{y}$ for (D).



Note: (P) infeasible does not imply that (D) is unbounded. We can create instances where both (P) and (D) are infeasible—see below. Here, both (P) and (D) are obviously infeasible.

$$
\begin{array}{lll}
\max & z = x_1 + 2x_2 & \\
\text{s.t.} & x_1 + x_2 = 1 & y_1 \text{ urs} \\
(P) & 2x_1 + 2x_2 = 3 & y_2 \text{ urs} \\
 & x_1, x_2 \text{ urs} & \\
 & = \quad = &
\end{array}
\qquad
\begin{array}{ll}
\min & w = y_1 + 3y_2 \\
\text{s.t.} & y_1 + 2y_2 = 1 \\
 & y_1 + 2y_2 = 2 \\
 & y_1, y_2 \text{ urs}
\end{array}
\quad (D)
$$

The Dual Theorem Let $\bar{x}_B$ be the optimal basic solution to (P), B be the basis matrix, $\bar{c}_B$ the basic cost vector. Then $\bar{y}^T = \bar{c}_B^T B^{-1}$ is optimal for (D), and $z^* = w^* = \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}^T} \bar{b} = \bar{y}^T \bar{b}$.

While we were working with (P) as a normal max-LP and hence (D) as a normal min-LP, this result holds even when (P) is a general max-LP.

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	$-\bar{c}_B^T$	$-\bar{c}_N^T$	0
0	B	N	$\bar{b}$

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	0	$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N$	$\bar{c}_B^T B^{-1} \bar{b}$
0	I	$B^{-1} N$	$B^{-1} \bar{b}$

IDEA Start with $\bar{y}^T = \bar{c}_B^T B^{-1}$.
Check that $\bar{y}$ is feasible for (D), and then check that $w = \bar{y}^T \bar{b} = z$

Optimality of the tableau for (P): $-\bar{c}^T + \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}^T} A \geq \bar{0}$

(all numbers under $\bar{x}$ in Row-0, or z-Row, are ≥ 0 for optimality)

Setting $\bar{y}^T = \bar{c}_B^T B^{-1} \Rightarrow -\bar{c}^T + \bar{y}^T A \geq \bar{0} \Rightarrow A^T \bar{y} \geq \bar{c}$

Consider the optimality criteria for slack variables:

$$\bar{0}^T + \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}^T}(I) \geq \bar{0}^T \Rightarrow \bar{y}^T \geq \bar{0}^T \Rightarrow \bar{y} \geq \bar{0}$$

Hence $\bar{y}^T = \bar{c}_B^T B^{-1}$ is feasible for (D).

But $w^* = \bar{b}^T \bar{y} = \bar{y}^T \bar{b} = \bar{c}_B^T B^{-1} \bar{b} = z^*$ in the optimal primal tableau. Hence by Lemma 2 (strong duality), $\bar{y}$ is optimal for (D).

Implication We can read off the optimal dual solution from the optimal primal tableau.

The Row-0 (or z-Row) in the optimal tableau is

$-\bar{c}^T + \bar{c}_B^T B^{-1} A$ (in general),

$-\bar{c}_B^T + \bar{c}_B^T B^{-1} B = \bar{0}$ for $\bar{x}_B$,

$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N$ for $\bar{x}_N$, and in particular,

$\bar{0} + \underbrace{\bar{c}_B^T}_{\bar{y}} B^{-1} I$ for slack variables

Hence we can read off the optimal $\bar{y}$ under slack columns for a normal max-LP, and more generally as follows.

Expressions in Row-0:

$\bar{0} + \bar{c}_B^T B^{-1} I$ for slack variables

$\bar{0} + \bar{c}_B^T B^{-1}(-I)$ for excess variables,

$M + \bar{c}_B^T B^{-1}(I)$ for artificial variables.

Hence, the optimal value of y_i (dual variable for constraint i) in a max-LP:

constraint i is $\leq$: coefficient of s_i in Row-0

constraint i is $\geq$: $-$(coefficient of e_i in Row-0)

constraint i is $=$: (coefficient of a_i in Row-0) $-$ M

Illustration

$$\begin{aligned} \max \quad & 30x_1 + 25x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 7 \quad y_1 \geq 0 \\ & 4x_1 + 10x_2 \leq 40 \quad y_2 \geq 0 \\ & 10x_1 \geq 30 \quad y_3 \leq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\geq$ $\geq$

$$\begin{aligned} \min \ & w = 7y_1 + 40y_2 + 30y_3 \\ \text{s.t.} \ & y_1 + 4y_2 + 10y_3 \geq 30 \\ (D) \ & y_1 + 10y_2 \geq 25 \\ & y_1, y_2 \geq 0, \ y_3 \leq 0 \end{aligned}$$

Optimal tableau (from Lecture 19):

BV	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	0	5	30	0	0	M	210
e_3	0	0	1	1	0	1	−1	4
s_2	0	0	6	−4	1	0	0	12
x_1	0	1	1	1	0	0	0	7

From the optimal tableau, $y_1 = 30$, $y_2 = 0$, $y_3 = 0$ is optimal for (D).
Note that $w = 7 \times 30 = 210 = z^*$, and hence must indeed be optimal.

Notice how we could read off the y_3 value from under either the e_3 or a_3 column here.

$$y_3 = -(\text{coefficient in Row-0 under } e_3) = 0$$

$$y_3 = (\text{Coefficient in Row-0 under } a_3) - M = M - M = 0.$$

Illustration on Farmer Jones LP

$$\begin{aligned} \max\ z &= 30x_1 + 100x_2 \\ \text{s.t.}\quad x_1 + x_2 &\le 7 \quad y_1 \\ 4x_1 + 10x_2 &\le 40 \quad y_2 \\ 10x_1 &\ge 30 \quad y_3 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} \min\ w &= 7y_1 + 40y_2 + 30y_3 \\ \text{s.t.}\quad y_1 + 4y_2 + 10y_3 &\ge 30 \qquad (D) \\ y_1 + 10y_2 &\ge 100 \\ y_1 \ge 0,\ y_2 \ge 0,\ y_3 &\le 0 \end{aligned}$$

$x_1 = 3, x_2 = 2.8, s_1 = 1.2$ with $z^* = 370$.

See the Matlab session on the course web page. We use $BV = \{x_1, x_2, s_1\}$ to directly compute the optimal tableau.

Here is the optimal tableau:

$M-1$ = 10,000

z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
1	0	0	0	10	1	9999	370
0	1	0	0	0	-1/10	1/10	3
0	0	1	0	1/10	1/25	-1/25	14/5 = 2.8
0	0	0	1	-1/10	3/50	-3/50	6/5 = 1.2

$y_1 = (\text{coefficient of } s_1 \text{ in Row-0}) = 0.$

$y_2 = (\text{coefficient of } s_2 \text{ in Row-0}) = 10.$

$y_3 = -(\text{coefficient of } e_3 \text{ in Row-0}) = -10.$

Also, $y_3 = (\text{coefficient of } a_3 \text{ in Row-0}) - M = 9999 - \overset{M}{10{,}000} = -1.$

Shadow Price of constraint $i = y_i$ (optimal value of dual variable)

Change $b_i \leftarrow b_i + \Delta$, find new optimal solution $(\bar{x}_B^{\Delta})$ assuming the basis remains same. Then find new optimal z^*, z^*_Δ, and write $z^*_\Delta = z^* + p_i\Delta$. Then p_i is the shadow price.

By strong duality, $z^* = w^*$, and $z^*_\Delta = w^*_\Delta$.

optimal dual objective function value with $b_i+\Delta$

But $w^* = b_1y_1 + \cdots b_iy_i + \cdots + b_my_m$ and $w^*_\Delta = b_1y_1 + \cdots + (b_i+\Delta)y_i + \cdots + b_my_m$

$$\Rightarrow w^*_\Delta = \underbrace{b_1y_1 + \cdots + b_iy_i + \cdots + b_my_m}_{w^*} + y_i\Delta = w^* + y_i\Delta$$

$$\Rightarrow z^*_\Delta = w^*_\Delta = w^* + y_i\Delta = z^* + y_i\Delta.$$

Hence $p_i = y_i$, i.e., shadow price of constraint $i = y_i$.

Here, $y_1 = 0$, $y_2 = 10$, $y_3 = -1$. So, shadow price of (land) $= 0$, that of (labor hours) $= 10$, and that of (min-corn) $= -1$.

If the min-corn requirement goes up by 1 unit, i.e., from 30 to 31, the rhs as written here would go up from 30 to 31, and the z^* value will decrease from 370 to $370 + (-1)\cdot 1 = 369$.

(See the course web page for the AMPL session).

MATH 364: Lecture 24 (11/07/2024)

Today: * complementary slackness conditions (CSC)

Complementary Slackness Conditions (CSCs)

$$(P)\quad \begin{array}{ll} \max & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}\le\bar{b} \quad \bar{y}\geqslant\bar{0} \\ & \bar{x}\geqslant\bar{0} \\ & \geqslant \end{array} \qquad\qquad \begin{array}{ll} \min & w=\bar{b}^T\bar{y} \\ \text{s.t.} & A^T\bar{y}\geqslant\bar{c} \\ & \bar{y}\geqslant\bar{0} \end{array}\quad (D)$$

Let $\bar{x}$ and $\bar{y}$ be feasible for (P) and (D), respectively.

Optimality conditions

- $\rightarrow$ $-\bar{c}^T+\bar{c}_B^TB^{-1}A\geqslant\bar{0}^T$ (Row-0 in tableau simplex)
- $\rightarrow$ If $z=\bar{c}^T\bar{x}=\bar{b}^T\bar{y}=w$, then $\bar{x}$ and $\bar{y}$ are optimal for (P) and (D), respectively
- $\rightarrow$ CSCs

Naturally, all three optimality conditions are equivalent.

CSCs

$$\bar{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix},\quad \bar{y}=\begin{bmatrix}y_1\\ \vdots\\ y_m\end{bmatrix}.$$

To convert (P) and (D) to standard form, we use slack variables $s_1,\dots,s_m$ in (P), and excess variables $e_1,e_2,\dots,e_n$ in (D). Equivalently, let

$$\bar{s}=\begin{bmatrix}s_1\\ s_2\\ \vdots\\ s_m\end{bmatrix}\quad\text{and}\quad \bar{e}=\begin{bmatrix}e_1\\ e_2\\ \vdots\\ e_n\end{bmatrix}$$

<u>CSCs</u> $\bar{x}$ and $\bar{y}$ are optimal for (P) and (D), respectively if and only if

$$s_i y_i = 0 \quad \text{for } i=1,\dots,m$$

$$\text{and} \quad e_j x_j = 0 \quad \text{for } j=1,\dots,n$$

In words, the product of slack/excess variable and the corresponding dual variable is zero at optimality. Hence, at least one of them is zero!

Equivalently, if a constraint is non-binding, the corresponding variable in the complementary (i.e., dual) problem must be zero at optimality.

Recall: Farmer Jones LP:

$$\begin{array}{lrcll} \max & z = 30x_1 + 100x_2 & & & \\ \text{s.t.} & x_1 + x_2 & \le & 7 & y_1 \\ & 4x_1 + 10x_2 & \le & 40 & y_2 \\ & x_1 & \geq & 3 & y_3 \\ & x_1, x_2 & \geq & 0 & \end{array}$$

$$\begin{array}{lrcll} \min & w = 7y_1 + 40y_2 + 3y_3 & & & \\ \text{s.t.} & y_1 + 4y_2 + y_3 & \geq & 30 & e_1 \\ & y_1 + 10y_2 & \geq & 100 & e_2 \\ & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \le 0 & & & \end{array}$$

$x_1 = 3,\ x_2 = 2.8,\ z^* = 370$

$s_1 = 1.2,\ s_2 = 0,\ e_3 = 0$

$s_1 \underset{\overset{\|}{0}}{y_1} = 0, \quad \underset{\overset{\|}{0}}{s_2} y_2 = 0, \quad \underset{\overset{\|}{0}}{e_3} y_3 = 0$ ✓

$y_1 = 0,\ y_2 = 10,\ y_3 = -10,\ w^* = 370$

$e_1 = 0,\ e_2 = 0$

$\underset{\overset{\|}{0}}{e_1} x_1 = 0, \quad \underset{\overset{\|}{0}}{e_2} x_2 = 0$ ✓

CSCs hold for any pair of (P)/(D) LPs, not just for normal LPs.

Using Complementary Slackness

For the given LP, solve its dual LP, and then use CSCs to solve the original LP.

$$\text{max } z = 5x_1 + 3x_2 + x_3$$

$$(P)\quad \text{s.t.}\quad \begin{array}{ll} 2x_1 + x_2 + x_3 \le 6 & y_1 \ge 0 \; s_1 \\ x_1 + 2x_2 + x_3 \le 7 & y_2 \ge 0 \; s_2 \\ x_1,\ x_2,\ x_3 \ge 0 & \\ \ge \quad\ \ge \quad\ \ge & \end{array}$$

$$\text{min } w = 6y_1 + 7y_2$$

$$(D)\quad \text{s.t.}\quad \begin{array}{ll} 2y_1 + y_2 \ge 5 & e_1 \\ y_1 + 2y_2 \ge 3 & e_2 \\ y_1 + y_2 \ge 1 & e_3 \\ y_1,\ y_2 \ge 0 & \end{array}$$

Optimal solution (D): $y_1 = \frac{7}{3}$. $y_2 = \frac{1}{3}$, $w^* = \frac{49}{3}$.

In (D), constraint 1 : $2\left(\frac{7}{3}\right) + \frac{1}{3} = 5 \Rightarrow e_1 = 0$

$$\left(\frac{7}{3}\right) + 2\left(\frac{1}{3}\right) = 3 \Rightarrow e_2 = 0$$

$$\left(\frac{7}{3}\right) + \left(\frac{1}{3}\right) = 8/3 \Rightarrow e_3 = 5/3$$

By CSCs, $x_3 = 0$ at optimality, as $e_3 x_3 = 0$.
Also, as $y_1 > 0$ and $y_2 > 0$, we get $s_1 = 0$, $s_2 = 0$ (as $s_i y_i = 0$).

$$\left.\begin{array}{l} 2x_1 + x_2 + \overset{=0}{\cancel{x_3}} \overset{=}{\cancel{\le}} 6 \\ x_1 + 2x_2 + \overset{=0}{\cancel{x_3}} \underset{=}{\cancel{\le}} 7 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2x_1 + x_2 = 6 \\ x_1 + 2x_2 = 7 \end{array}\right\} \quad x_2 = 8/3,\ x_1 = 5/3$$

So $x_1 = 5/3$, $x_2 = 8/3$ is optimal.

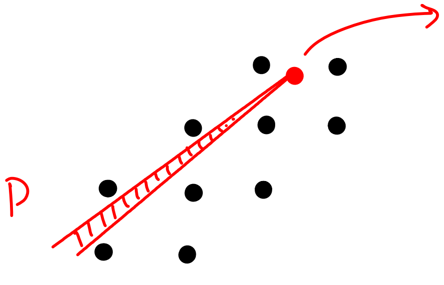
Indeed, $z^* = 5\left(5/3\right) + 3\left(8/3\right) = \frac{49}{3} = w^*$.

Of course, the use of CSCs is more widespread than indicated by the above toy example. There are classes of optimization algorithms based on each type of optimality conditions. The ones based on CSCs start with pairs of solutions $\bar{x}$ and $\bar{y}$ that do not satisfy all CSCs, but may be satisfy feasibility for (P) and (D), and then progressively satisfy the CSCs. The economic interpretation is also quite important.

In the next lecture, we will talk about integer programming!

Q: Could we just round the continuous solution?

Might not even be feasible!!



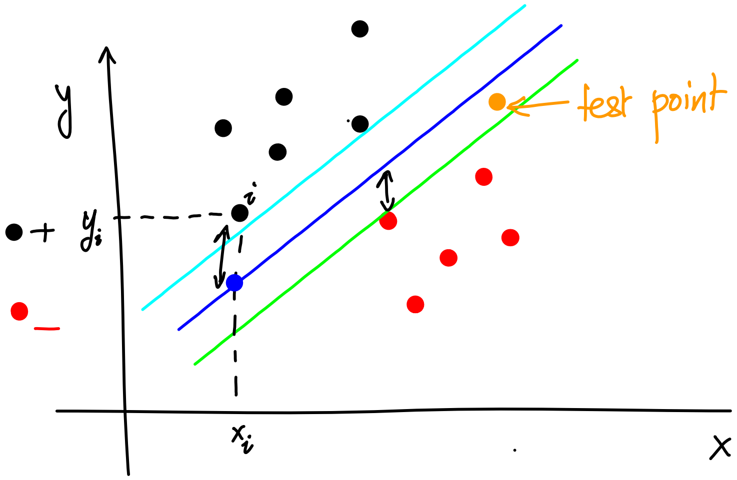
rounding this corner point in any fashion puts us out of the feasible region.

MATH 364: Lecture 25 (11/12/2024)

Today: * about the project
* IP formulations

Project: Motivated by support vector machines (SVM)

In (linear) regression, you fit a straight line that best represents the given set of data points. This line minimizes the sum of squared errors. This is a non-linear objective function.



Here, we want to find a "separating line" which separates the +1 and −1 points as "widely" as possible.

variables

$$y_i = \bar{w}^T \bar{x}^i + w_0 \qquad y_i, \bar{x}^i \text{ are data here}$$

Data has $y_i \in \{+1,-1\}$, so, ideally, we want

$$\bar{w}^T \bar{x}^i + w_0 = +1 \quad \text{for } y_i = +1 \text{ instances.}$$

In fact, we want $\bar{w}^T \bar{x} + w_0 = 1 + \varepsilon_i$, $\varepsilon_i > 0$ if possible.

Solve an LP to identify $(\bar{w}, w_0)$. Then use this predictor to predict on the test set.

There are two LP models suggested in the project description. You may still have to modify the models to get best results. Similarly, you may have to play around with the parameters, e.g., upper bounds on some of the variable values to get the best results.

Getting the data read into AMPL

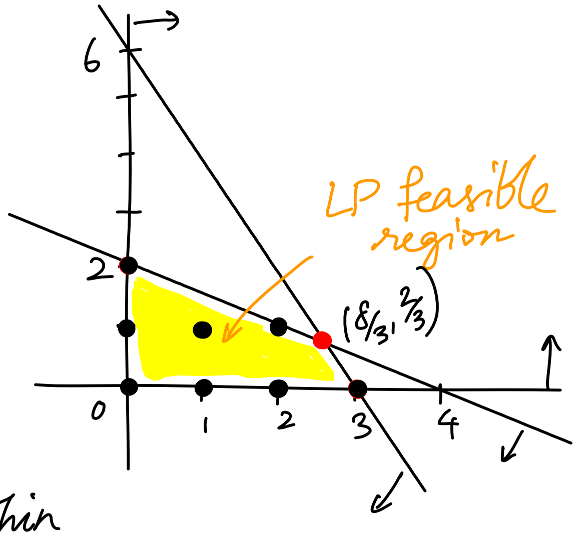
Since the data is given in text files (TrainingSet.txt and TestSet.txt), one could read in these data files directly into AMPL using the **read** command.

See the course web page for an example.

Integer (Linear) Programming (IP)

An LP in which each variable is restricted to take only integer values is called pure integer program, or integer program (IP) by default.

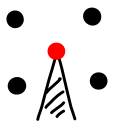
e.g., max $z = 3x_1 + 2x_2$
s.t.
$x_1 + 2x_2 \leq 4$ (IP)
(LP) $2x_1 + x_2 \leq 6$
$x_1, x_2 \geqslant 0$, x_1, x_2 integer



The feasible set here consists of the 8 points with integer coordinates within P, the LP feasible region.

The LP optimal solution is at $(\frac{8}{3}, \frac{2}{3})$, while the IP optimal solution is at $(3,0)$ $(z^* = 9$ at $(3,0))$.

Rounding the LP optimal solution works here, but might not work always.



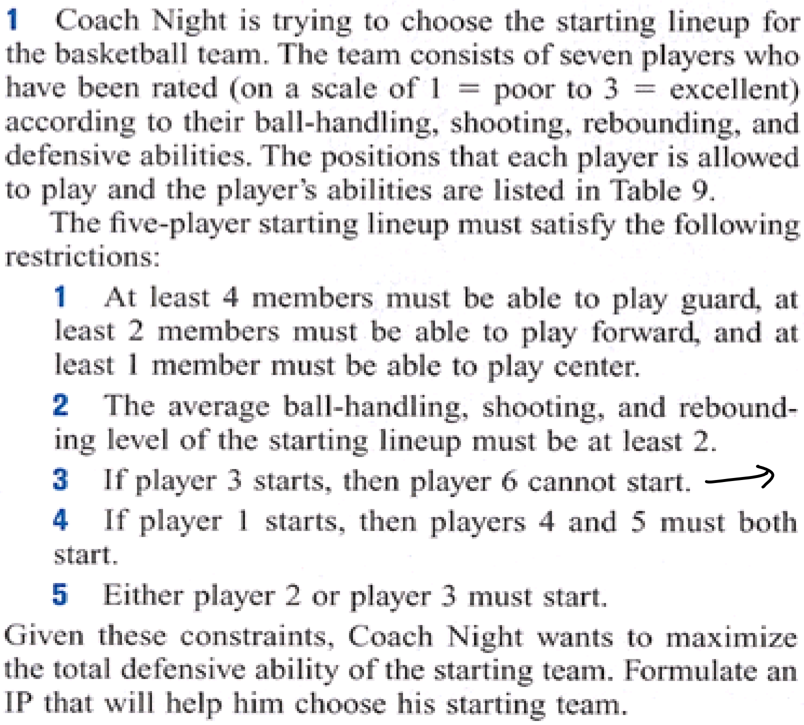
All possible roundings are IP-infeasible here!

An LP in which a subset of variables is restricted to take integer values is called mixed integer program (MIP).

If the integer variables can take values only in $\{0,1\}$, then the IP is called a Binary IP (BIP).

If we drop the integer restriction from an IP or MIP, we get its LP-relaxation.

IP Formulations



1 Coach Night is trying to choose the starting lineup for the basketball team. The team consists of seven players who have been rated (on a scale of 1 = poor to 3 = excellent) according to their ball-handling, shooting, rebounding, and defensive abilities. The positions that each player is allowed to play and the player's abilities are listed in Table 9.

The five-player starting lineup must satisfy the following restrictions:

1 At least 4 members must be able to play guard, at least 2 members must be able to play forward, and at least 1 member must be able to play center.

2 The average ball-handling, shooting, and rebounding level of the starting lineup must be at least 2.

3 If player 3 starts, then player 6 cannot start. → both 3 & 6 cannot start together

4 If player 1 starts, then players 4 and 5 must both start.

5 Either player 2 or player 3 must start.

Given these constraints, Coach Night wants to maximize the total defensive ability of the starting team. Formulate an IP that will help him choose his starting team.

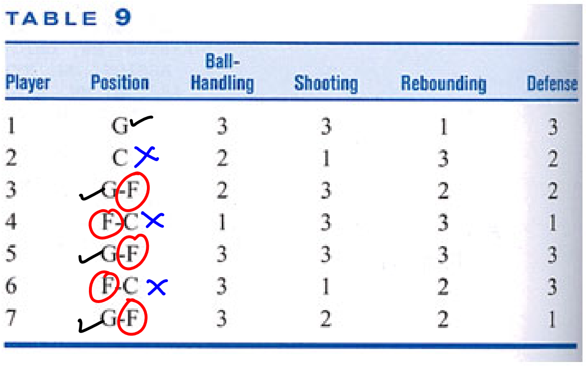


TABLE 9

Player	Position	Ball-Handling	Shooting	Rebounding	Defense
1	G✓	3	3	1	3
2	C ×	2	1	3	2
3	✓G-F	2	3	2	2
4	F-C ×	1	3	3	1
5	✓G-F	3	3	3	3
6	F-C ×	3	1	2	3
7	✓G-F	3	2	2	1

Decisions: For each player, do they start or not: YES/NO.

d.v.s : $x_j = \begin{cases} 1 & \text{if player } j \text{ starts} \\ 0 & \text{otherwise} \end{cases}$, $j=1,\dots,7$

Constraints

0. $\sum_{j=1}^{7} x_j = 5$ (five starters)

1. $x_1 + x_3 + x_5 + x_7 \geq 4$ (guards)

$x_3 + x_4 + x_5 + x_6 + x_7 \geq 2$ (forwards)

$x_2 + x_4 + x_6 \geq 1$ (center)

2. $$\frac{3x_1+2x_2+2x_3+x_4+3x_5+3x_6+3x_7}{5 \ (\text{or } \sum_j x_j)} \geq 2$$ (avg ball-handling)

$$3x_1 + \cdots + 2x_7 \geq 5 \times 2$$ (avg. shooting)

$$x_1 + \cdots + 2x_7 \geq 5 \times 2$$ (avg. rebounding)

3. $x_3 + x_6 \leq 1$ (3 and 6 cannot both start)

or $x_6 \leq 1 - x_3$ (if 3 starts, 6 cannot)

$x_3 + x_6 = 1$ insists exactly one of 3 and 6 has to start

4. $x_4 \geq x_1$ (if 1 starts, 4 and 5 must also start)

$x_5 \geq x_1$

One constraint:

~~$x_4 + x_5 \geq x_1 + 1$~~ ? as $x_1 = 0$ would still insist $x_4 + x_5 \geq 1$

But

$x_4 + x_5 \geq 2x_1$ works. $x_1 = 1 \Rightarrow x_4 + x_5 \geq 2$

$x_1 = 0 \Rightarrow x_4 + x_5 \geq 0$, which is redundant

$x_4 = x_1$, $x_5 = x_1$ insists 1, 4, and 5 all start together or all not start.

5. $x_2 + x_3 \geq 1$ (2 or 3 must start, or both)

$x_2 + x_3 = 1$ → exactly one of them starts

Objective function:

$$\max\ z = 3x_1 + 2x_2 + \cdots + x_7 \quad \text{(total defense ability)}$$

If we were maximizing the average defense ability, we would set the objective function as

$$\max\ z = \frac{1}{5}\left(3x_1 + 2x_2 + \cdots + x_7\right) \quad \text{(avg. defense ability)}$$

MATH 364 : Lecture 26 (11/14/2024)

Today: * Fixed charge IP
* either-or statements

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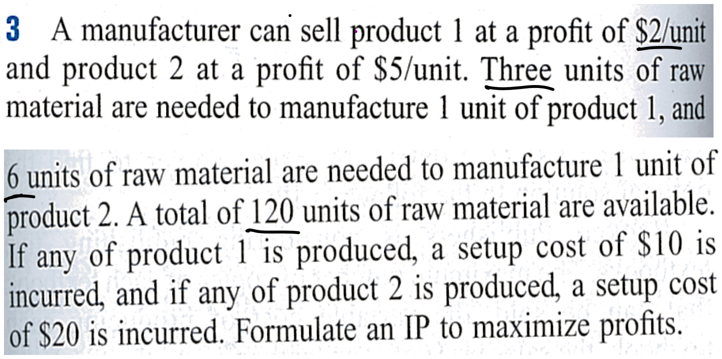
2. Fixed charge (or setup cost) problem



3 A manufacturer can sell product 1 at a profit of $2/unit and product 2 at a profit of $5/unit. Three units of raw material are needed to manufacture 1 unit of product 1, and 6 units of raw material are needed to manufacture 1 unit of product 2. A total of 120 units of raw material are available. If any of product 1 is produced, a setup cost of $10 is incurred, and if any of product 2 is produced, a setup cost of $20 is incurred. Formulate an IP to maximize profits.

Decisions: ① produce any of product 1,2 at all? YES/NO
② how many of each to produce?

d.v.'s

$$\text{let } y_i = \begin{cases} 1 & \text{if we make any of product } i \\ 0, & \text{otherwise} \end{cases}, \quad i=1,2$$

and x_i = # units of product i made, $i=1,2$.

We need $y_i \overset{\text{"element of"}}{\in} \{0,1\}$, $x_i \geq 0$

So, $$y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}, \quad i=1,2.$$ This is just the definition (or description) of y_i.

We need to enforce the relationship between y_i and x_i using linear inequalities.

Constraints

$$3x_1 + 6x_2 \leq 120 \quad \text{(raw matl.)}$$

$$x_1 \leq M_1 y_1, \quad M_1 > 0$$
$$x_2 \leq M_2 y_2, \quad M_2 > 0$$
(M_i large, positive).

$M_1 = \frac{120}{3} = 40$ and $M_2 = \frac{120}{6} = 20$ work here.

If no such info is known, can use $M_1 = M_2 = 10^5$, say.

Let's see why these constraints are correct:

$$x_1 \leq M_1 y_1$$

If $x_1 > 0$, the only way this constraint will hold is with $y_1 = 1$.
Hence $y_1 = 1$ when $x_1 > 0$.
If $x_1 = 0$, the constraint holds with $y_1 = 0$ or with $y_1 = 1$.
In particular, $x_1 = 0$ does not force $y_1 = 0$ here. → We will have the objective function doing this forcing!

<u>Objective function</u> : $\max z = \underbrace{2x_1 + 5x_2}_{\text{revenue}} - \underbrace{10y_1 - 20y_2}_{\text{costs}}$ (profit)

The coefficient of y_1 is -10 in a max obj. fn, hence y_1 is forced to 0 when possible. Hence when $x_1 = 0$, we get $y_1 = 0$ in the optimal solution.

The whole MIP:

$$\begin{array}{lrll} \max & z = 2x_1 + 5x_2 - 10y_1 - 20y_2 & & \text{(profit)} \\ \text{s.t.} & 3x_1 + 6x_2 & \leq 120 & \text{(raw matl.)} \\ & x_1 & \leq 40y_1 & \text{(forcing const 1)} \\ & x_2 & \leq 20y_2 & \text{(" " 2)} \\ & & & \\ & x_1, x_2 \geq 0, & y_1, y_2 \in \{0,1\}. & \end{array}$$

$x_1 \leq M_1 y_1$: with $y_1 = 1$, this constraint specifies an upper bound on x_1; hence we use the bound as suggested by raw material availability.

In general, we write $\sum_j \alpha_j x_j \leq M y_i \xrightarrow{} 0,1$.

When $y_i = 1$, the constraint is $\sum_j \alpha_j x_j \leq M$.

We use the smallest M that makes sense from data. If not able to estimate, use $M = 10^5$, say, or some similar large number.

See the course web page for AMPL files.

When should we insist on integrality?

1. Should we build dorm on campus? YES/NO
2. How many rooms to include?

We do need binary variable here...

Could get away w/ a continuous variable.

say, $x = 234.6$; Choosing $x = 234$ or 235 may not make a huge difference.

Either-Or constraint

$$f(x_1,\dots,x_n) \le 0 \text{ ——— } (1)$$
$$g(x_1,\dots,x_n) \le 0 \text{ ——— } (2)$$

Model: Either (1) or (2) must hold.

Use an extra binary var:

Let $y = \begin{cases} 1 & \text{if (2) holds, and} \\ 0 & \text{otherwise} \Rightarrow \text{(1) holds} \end{cases}$

Assume there is $M > 0$ large enough such that $f(x_1,\dots,x_n) \le M$ and $g(x_1,\dots,x_n) \le M$ always hold.

Model:

$$\boxed{\begin{array}{ll} f(x_1,\dots,x_n) \le My & \text{——— } (3) \\ g(x_1,\dots,x_n) \le M(1-y) & \text{——— } (4) \\ y \in \{0,1\} & \text{——— } (5) \end{array}}$$

If $y=0$, (3) $\Rightarrow \underbrace{f(\cdot) \le 0}_{(1)}$ and (4) $\Rightarrow \underbrace{g(\cdot) \le M}_{\text{redundant}}$

If $y=1$ (3) $\Rightarrow \underbrace{f(\cdot) \le M}_{\text{redundant}}$ and (4) $\Rightarrow \underbrace{g(\cdot) \le 0}_{(2)}$.

Notice $f(\cdot) \le M$ is always true, and is **not** implying (2) holds.

In the basketball starting line-up problem, we had

(5) either <u>player 2</u> or <u>3</u> must start.

We wrote $x_2 + x_3 \geq 1$ ← does allow both to start

(from logic).

$x_2 = 1$ $\quad$ $x_3 = 1$

or $x_2 \geq 1$, $\quad$ $x_3 \geq 1$

So, we want

$$1 - x_2 \leq 0$$

or

$$1 - x_3 \leq 0$$

<u>M=1</u> works here. So we can write

$$\boxed{\begin{aligned} 1 - x_2 &\leq y \\ 1 - x_3 &\leq 1 - y \\ y &\in \{0,1\} \end{aligned}}$$

$1 - x_2 \leq 1$ as $x_2 \in \{0,1\}$

When $y=1$, we get $1 - x_2 \leq 1 \Rightarrow x_2 \geq 0$ ✓

$1 - x_3 \leq 0 \Rightarrow x_3 \geq 1$

Adding these two inequalities, we get $\cancel{1} - x_2 + 1 - x_3 \leq \cancel{1}$

i.e., $\quad x_2 + x_3 \geq 1$

If we want to allow both (1) and (2) to hold together, we can write

$$f(\cdot) \leq M(1 - y_f)$$
$$g(\cdot) \leq M(1 - y_g)$$
$$y_f + y_g \geq 1$$
$$y_f, y_g \in \{0,1\}$$

Note: If we write $y_f + y_g = 1$, we get the previous model.

If we have more than two alternatives:

$$\left.\begin{array}{l} f_1(\cdot) \leq 0 \\ \vdots \qquad \vdots \\ f_k(\cdot) \leq 0 \end{array}\right\} \text{at least one alternative holds}$$

We can write

$$\begin{array}{l} f_1(\cdot) \leq M(1-y_1) \\ \vdots \qquad\qquad \vdots \\ f_k(\cdot) \leq M(1-y_k) \\ y_1 + y_2 + \cdots + y_k \geq 1 \\ y_i \in \{0,1\} \end{array}$$

MATH 364: Lecture 27 (11/19/2024)

Today: * project submission details
* if-then statements as MIPs

Project * Solve LPs in AMPL;

- LP would give you w_0 and $\bar{w} = [w_1 \ldots w_n]^T$ weights.
- you still need to come up with a rule to convert the $y_i = \bar{w}^T \bar{x}_i + w_0$ for the traning set instances to +1/-1. Say, the y_i values lie between -4050 and +8623. Pick a cutoff, say δ, in this range, and make the rule that

  if $y_i \geqslant \delta$, assign +1
  else assign -1.

  Choice of δ is part of training. Pick δ so that as many (out 90) instances in the training set are assigned correctly.

- Repeat the prediction computations for test set: compute $y_i = \bar{w}^T \bar{x}_i + w_0$ for $i = 1, \ldots, 10$. Then apply the same δ to convert y_i to +1/-1.

- you're welcome to do other calculations outside AMPL (except LP part).

Report (≤ 4 pages)

- (Do **not** include the model file in the report — submit them separately ...)
- Describe which LPs you used.
- Specify ranges of values for parameters you tried. (provide brief justifiction).
- report the accuracy on the test set (how many out of 10 you predicted correctly).
- comment on why you got these results.

Submit all files including report, etc., as a one compressed folder (e.g., zip, ~~tar-gzipped~~, etc.).

→ follow instructions given in the project.

If-Then constraints

Recall: either-or constraints: $f(\cdot) \leq 0$ —— (1)
$g(\cdot) \leq 0$ —— (2)

Either (1) or (2) should hold:

$$\boxed{\begin{array}{c} f(\cdot) \leq My \\ g(\cdot) \leq M(1-y) \\ y \in \{0,1\} \end{array}}$$

If-then constraints

$-g(x_1,\dots,x_n) \leq 0$

if $f(x_1,\dots,x_n) > 0$ then $g(x_1,\dots,x_n) \geq 0$

If $f(\cdot) > 0$ then we want $g(\cdot) \geq 0$ to hold. But $f(\cdot) > 0$ means $f(\cdot) \leq 0$ does **not** hold. We convert the input statement to an equivalent either-or statement.

$A \Rightarrow B$ or (if A then B) $\equiv$ $\underline{\text{not } A}$ or B

"implies" "equivalent to" negation or opposite of A

$\equiv$ either $f(x_1,\dots,x_n) \leq 0$ or $-g(x_1,\dots,x_n) \leq 0$.

$$\boxed{\begin{array}{c} f(x_1,\dots,x_n) \leq My \\ -g(x_1,\dots,x_n) \leq M(1-y) \\ y \in \{0,1\} \end{array}}$$

27.4

12 A company is considering opening warehouses in four cities: New York, Los Angeles, Chicago, and Atlanta. Each warehouse can ship 100 units per week. The weekly fixed cost of keeping each warehouse open is \$400 for New York, \$500 for Los Angeles, \$300 for Chicago, and \$150 for Atlanta. Region 1 of the country requires 80 units per week, region 2 requires 70 units per week, and region 3 requires 40 units per week. The costs (including production and shipping costs) of sending one unit from a plant to a region are shown in Table 11. We want to meet weekly demands at minimum cost, subject to the preceding information and the following restrictions:

f_i

1. If the New York warehouse is opened, then the Los Angeles warehouse must be opened.
2. At most two warehouses can be opened.
3. Either the Atlanta or the Los Angeles warehouse must be opened.

d_j → demands.

Formulate an IP that can be used to minimize the weekly costs of meeting demand.

TABLE 11

	To (\$)		
From	Region 1	Region 2	Region 3
New York	20	40	50
Los Angeles	48	15	26
Chicago	26	35	18
Atlanta	24	50	35

c_{ij}

Decisions

1. should a warehouse be opened in City i or not, $i = N, L, C, A$ (for NY, LA, Ch, At).

2. How many units to ship from warehouse i to region j, $i = N, L, C, A$, $j = 1, 2, 3$.

d.V.'s

Let $y_i = \begin{cases} 1 & \text{if warehouse opened in city } i \\ 0 & \text{otherwise} \end{cases}$, $i = N, L, C, A$

and x_{ij} = # units shipped from warehouse i to region j, $i = N, L, C, A$, $j = 1, 2, 3$. (≥ 0).

Let f_i = fixed charge at city i (\$400 for NY, ...) and

c_{ij} = unit shipping charge from city i to region j (in Table)

d_j = demand in region j. $j = 1,2,3$ (80, 70, 40)

for compact representation of MIP model

Objective function

$$\min \; z = \sum_{i=N,L,C,A} f_i y_i + \sum_{i=N,L,C,A} \sum_{j=1}^{3} c_{ij} x_{ij} \qquad \text{(total cost)}$$

Constraints

$$\sum_{j=1}^{3} x_{ij} \leq \overset{M_i \text{ here}}{\overset{\downarrow}{100}}\, y_i \,, \quad i = N, L, C, A \qquad \text{(forcing constraints)}$$

Alternatively, we could split up these constraints:

$$x_{ij} \leq d_j y_i \,, \quad i = N, L, C, A, \quad j = 1, 2, 3 \qquad \text{(max shipping)}$$

$\to$ works, as $d_j \leq 100$ for each j here

But in this case, we need to write

$$\sum_{j=1}^{3} x_{ij} \leq 100 \,, \quad i = N, L, C, A \qquad \text{(max shipping)}$$

$$\sum_{i=N,L,C,A} x_{ij} \geq d_j \,, \quad j = 1, 2, 3 \qquad \text{(Region } j \text{ demand)}$$

Logical constraints

1. If NY then LA $\equiv$ if $\underbrace{\overbrace{y_N}^{f(\cdot)} > 0}_{\equiv (y_N = 1)}$ then $\underbrace{y_L \geq 1}_{\equiv (y_L = 1)} \equiv \overbrace{y_L - 1}^{g(\cdot)} \geq 0$

$$y_N \leq z_1$$
$$-(y_L - 1) \leq 1 - z_1$$
$$z_1 \in \{0, 1\}$$

Used $M = 1$ here; $-g(\cdot) \equiv -(y_L - 1)$
$= 1 - y_L$

if $y_N > 0$ then $y_L \geq 1$ $\equiv$ either $y_N \leq 0$ or $1 - y_L \leq 0$

$$\left.\begin{array}{l} y_N \leq z_1 \\ 1 - y_L \leq 1 - z_1 \end{array}\right\} \text{adding these statements gives } y_N \leq y_L$$

$$z_1 \in \{0,1\}$$

Alternatively, using direct logic, we could write

$y_L \geq y_N$ (if NY then LA)

2. $\sum_{i=N,L,C,A} y_i \leq 2$ (at most 2 warehouses open)

3. Either A or L : $y_A \geq 1$ or $y_L \geq 1$

$\equiv$ $1 - y_A \leq 0$ or $1 - y_L \leq 0$

$$\left.\begin{array}{l} 1 - y_A \leq z_3 \\ 1 - y_L \leq 1 - z_3 \end{array}\right\} y_A + y_L \geq 1$$

$$z_3 \in \{0,1\}$$

Or, using logic, $y_A + y_L \geq 1$

Variable restrictions

$y_i \in \{0,1\}$, $i = N,L,C,A$

$x_{ij} \geq 0$, $i = N,L,C,A$, $j = 1,2,3$

Could write
$1 \leq x \leq 4$
x integer

Next lecture: Model: $\underset{3}{x=1}$ or $\underset{8}{x=2}$ or $\underset{13}{x=3}$ or $\underset{21}{x=4}$

MATH 364: Lecture 28 (11/21/2024)

Today: * IP formulations

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1. Model: $x=1$ or $x=2$ or $x=3$ or $x=4$

Here: ($1\le x\le 4$, x integer) works. But what if the options were 15, 7, 23, 48 ?

Original statement is

$x\le 1$ and $x\ge 1$ OR $x\le 2$ and $x\ge 2$ OR $x\le 3$ and $x\ge 3$ OR $x\le 4$ and $x\ge 4$

Let $t_1\in\{0,1\}$ $t_2\in\{0,1\}$ $t_3\in\{0,1\}$ $t_4\in\{0,1\}$

$x-1\le 0$ OR $x-2\le 0$ OR $x-3\le 0$ OR $x-4\le 0$

$$x-1\le M(1-t_1)$$
$$x-2\le M(1-t_2)$$
$$x-3\le M(1-t_3)$$
$$x-4\le M(1-t_4)$$

$M=4$ works here

$$t_1+t_2+t_3+t_4=1$$
$$t_i\in\{0,1\},\ i=1,2,3,4$$

The second set of four alternatives can be modeled as follows:

$$\begin{aligned} -x+1 &\leq M(1-t_1) \\ -x+2 &\leq M(1-t_2) \\ -x+3 &\leq M(1-t_3) \\ -x+4 &\leq M(1-t_4) \end{aligned}$$

We already have

$$t_1+t_2+t_3+t_4=1$$

$$t_i \in \{0,1\},\ i=1,2,3,4.$$

Can put them all together:

$$\begin{aligned} x-1 &\leq M(1-t_1) & \qquad -x+1 &\leq M(1-t_1) \\ x-2 &\leq M(1-t_2) & -x+2 &\leq M(1-t_2) \\ x-3 &\leq M(1-t_3) & -x+3 &\leq M(1-t_3) \\ x-4 &\leq M(1-t_4) & -x+4 &\leq M(1-t_4) \end{aligned}$$

$$t_1+t_2+t_3+t_4 \overset{\geq}{=} 1$$

$$t_i \in \{0,1\},\ i=1,2,3,4.$$

→ would also work here → just that we'll never have more than one $t_i=1$ at the same time

In general, if you're not sure whether it is an XOR or inclusive OR, you could go either way.

28.3

2. Model the following statement using extra binary variables:

if $x \leq 2$ then $y \leq 3$, where $x, y \in \mathbb{Z}$ (integers)

$A \Rightarrow B \equiv \text{not} A \text{ OR } B$

We want alternatives expressed in $f(\cdot) \leq 0$, $g(\cdot) \leq 0$, etc. form.

Statement is equivalent to

either $x > 2$ or $y \leq 3$

$\equiv$ either $x \geq 3$ or $y \leq 3$, as $x \in \mathbb{Z}$

$\equiv$ either $-x+3 \leq 0$ or $y-3 \leq 0$

$$\boxed{\begin{aligned} -x+3 &\leq Mt \\ y-3 &\leq M(1-t) \\ t &\in \{0,1\} \end{aligned}}$$

We cannot estimate a value for M here — just leave as M.

XOR!

$$\boxed{\begin{aligned} -x+3 &\leq M(1-t_1) \\ y-3 &\leq M(1-t_2) \\ t_1+t_2 &\geq 1 \\ t_1, t_2 &\in \{0,1\} \end{aligned}}$$

This formulation allows either one or both alternatives to hold.

3. Model: if $x+2y>2$ holds, then either $2x+3y\le 5$ holds or $3x+4y\ge 4$ holds.

(Note: x, y are not assumed to be integers here.)

Statement is equivalent to

either $x+2y\le 2$ or $(2x+3y\le 5$ or $3x+4y\ge 4)$

$\equiv$ $x+2y-2\le 0$ or $2x+3y-5\le 0$ or $-3x-4y+4\le 0$

$$\boxed{\begin{aligned} x+2y-2 &\le M(1-t_1)\\ 2x+3y-5 &\le M(1-t_2)\\ -3x-4y+4 &\le M(1-t_3)\\ t_1+t_2+t_3 &\ge 1\\ t_1, t_2, t_3 &\in \{0,1\} \end{aligned}}$$

4. Model: if $|x| \leq 4$ then $|y| > 5$, where $x, y \in \mathbb{Z}$.

Statement is equivalent to

either $|x| > 4$ or $|y| > 5$

$\equiv$ either $|x| \geq 5$ or $|y| \geq 6$, as $x, y \in \mathbb{Z}$.

$$\equiv \text{ either } \begin{pmatrix} x \geq 5 \\ \text{or} \\ x \leq -5 \end{pmatrix} \text{ or } \begin{pmatrix} y \geq 6 \\ \text{or} \\ y \leq -6 \end{pmatrix}$$

$\equiv$ either $x \geq 5$ or $x \leq -5$ or $y \geq 6$ or $y \leq -6$

$\equiv$ either $-x+5 \leq 0$ or $x+5 \leq 0$ or $-y+6 \leq 0$ or $y+6 \leq 0$

$$\boxed{\begin{array}{c} -x+5 \leq M(1-t_1) \\ x+5 \leq M(1-t_2) \\ -y+6 \leq M(1-t_3) \\ y+6 \leq M(1-t_4) \\ t_1+t_2+t_3+t_4 \overset{\geq}{\leq} 1 \\ t_i \in \{0,1\}, \quad i=1,2,3,4 \end{array}}$$

→ also works

MATH 364: Lecture 29 (12/03/2024)

Today: * Problems from Hw8
* Practice final

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* Final exam will be posted on Wed, Dec 11
* Due by 10 pm on Thu, Dec 12 by email.
* Limited open resource exam!
  - ✓ anything posted on course web page
  - ✓ Can use AMPL

Hint on AMPL implementation for project:

* Declare params in model file for both training and test sets.
* Solve LP on training set data, then use the solution to evaluate on test set data at the ampl: prompt.

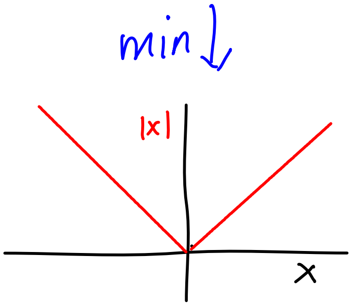
* No need to show any output from AMPL, or any model/data files in your report PDF. Include all AMPL files in your submission (separate from the report PDF).

Problems from Homework

Hw8. Problem 1

$\overset{\text{min}}{\cancel{\text{max}}}$ → could model as an LP if it's min!

$$\begin{aligned} \overset{\text{min}}{\cancel{\text{max}}}\quad & z = |3x_2-4x_1| \\ \text{s.t.}\quad & 6x_1+2x_2 \le 7 \\ & 3x_1+4x_2 \le 4 \\ & x_1, x_2 \ge 0 \end{aligned}$$



$|x| = \max\{x, -x\}$.

Recall: x urs $\longrightarrow$ $x^+ - x^-$, $x^+, x^- \ge 0$

So, one could possibly write

$$\cancel{\begin{aligned} \max\quad & z = z^+ + z^- \\ \text{s.t.}\quad & z^+ - z^- = 3x_2 - 4x_1 \\ & 6x_1 + 2x_2 \le 7 \\ & 3x_1 + 4x_2 \le 4 \\ & x_1, x_2, z^+, z^- \ge 0 \end{aligned}}$$

But this LP is unbounded.

Say $z = 3x_2 - 4x_1 = \alpha$ is the largest value it can take. Hence

$z^+ = \alpha$, $z^- = 0$ could be a valid solution.

Here, $z = z^+ + z^- = \alpha$, is what you want.

But, $z^+ = 23\alpha$, $z^- = 22\alpha$ gives $z^+ - z^- = \alpha$, while giving you $z^+ + z^- = 45\alpha >> \alpha$

More generally, max{max} or min{min} cannot be modeled as a linear program. min{max} or max{min} could be modeled.

Here, you have to consider two separate LPs

$$\begin{array}{lll} \max\ z^+ = 3x_2 - 4x_1 & & \min\ z^- = 3x_2 - 4x_1 \\ \text{s.t.}\ 6x_1 + 2x_2 \le 7 & \text{and} & \text{s.t.}\ 6x_1 + 2x_2 \le 7 \\ \quad 3x_1 + 4x_2 \le 4 & & \quad 3x_1 + 4x_2 \le 4 \\ \quad x_1, x_2 \geq 0 & & \quad x_1, x_2 \geq 0 \end{array}$$

Then take $\max \{ |z^{+*}|, |z^{-*}| \}$, and the corresponding optimal solution (x_1^*, x_2^*) as the answer.

Prob 2 (HW8)

Property holds at start:

$$\begin{array}{cc} x^+ & x^- \\ \hline c & -c \\ \hline a_{11} & -a_{11} \\ \vdots & \\ \frac{1}{\beta}(a_{i1} & -a_{i1}) \\ \vdots & \vdots \\ a_{m1} & -a_{m1} \\ \hline \end{array}$$

1. Scaling ERO: Divide by $\beta \neq 0$. (typically, $\beta > 0$).

$$\frac{1}{\beta}(a_{i1} \quad -a_{i1}) \rightarrow \frac{a_{i1}}{\beta} \quad -\frac{a_{i1}}{\beta} \ ✓$$

2. Replacement ERO: $R_i \leftarrow R_i + \alpha R_j$

$$a_{i1} \ -a_{i1} \longrightarrow a_{i1} + \alpha a_{j1} \qquad -a_{i1} + \alpha(-a_{j1})$$

$$\longrightarrow (a_{i1} + \alpha a_{j1}) \qquad -(a_{i1} + \alpha a_{j1}) \ ✓$$

i could be 0 here (for Row-0).

Prob 3, HW 8

(a) Let x_j replace x_ℓ, which is currently basic in Row-i.

Since x_j is entering (in a max-LP), its coefficient in Row-0 should be ≤ 0.

$$\begin{array}{rcc|l} & x_\ell & x_j & \\ \hline & 0 & -c_j & c_j > 0 \\ \hline & 0 & \vdots & \\ & \vdots & \vdots & \\ i \rightarrow & 1 & a_{ij} & > 0 \text{ (pivot)} \\ & 0 & \vdots & \\ & \vdots & \vdots & \\ & 0 & & \\ \hline \end{array}$$

We do $R_0 + \left(\frac{c_j}{a_{ij}}\right) R_i$ to zero out $-c_j$ (in Row-0) under x_j. Under x_ℓ in Row-0, we get

$$0 + \left(\frac{c_j}{a_{ij}}\right) 1 = \frac{c_j}{a_{ij}} > 0 \quad \left(\text{as both } c_j > 0 \text{ and } a_{ij} > 0\right).$$

could be $= 0$ if $c_j = 0$.

It is important to detail the effects of EROs in this fashion.

(b) Since coefficient of x_ℓ in Row-0 is $\overset{(\geq 0)}{> 0}$, it cannot enter back immediately into the basis of a max LP.

29.5

Practice Final Exam

6.

(P)
$$\begin{array}{lll} \min & z = 3x_1 + 3x_2 + 4x_3 & \\ \text{s.t.} & 4x_1 + 6x_2 + 3x_3 & \geq \ 7 \quad y_1 \geq 0 \\ & 3x_1 + x_2 + x_3 & \geq \ 3 \quad y_2 \geq 0 \\ & x_1, \ x_2, \ x_3 & \geq \ 0 \\ & \leq \quad \leq \quad \leq & \end{array}$$

(D)
$$\begin{array}{lll} \max & w = 7y_1 + 3y_2 & \\ \text{s.t.} & 4y_1 + 3y_2 \leq 3 & s_1 \\ & 6y_1 + y_2 \leq 3 & s_2 \\ & 3y_1 + y_2 \leq 4 & s_3 \geq 0 \\ & y_1, y_2 \geq 0 & \end{array}$$

As $s_3 = \frac{16}{7}$, $x_3 = 0$ (CSC).

From AMPL: $y_1 = \frac{3}{7}$, $y_2 = \frac{3}{7}$, $w^* = \frac{30}{7}$.

It would be efficient to use AMPL to solve (D) here. At the same time, you could verify the optimal solution for (P) as well!

$$3y_1 + y_2 = 3\left(\frac{3}{7}\right) + \left(\frac{3}{7}\right) = \frac{12}{7} = 4 - \frac{16}{7}. \text{ So } s_3 = \frac{16}{7}.$$

Since $s_3 > 0$, CSCs give $x_3 = 0$. (as $s_3 x_3 = 0$).

Also, since $y_1 > 0$ and $y_2 > 0$, CSCs give $s_1 = s_2 = 0$ $(s_i y_i = 0)$.

Hence in (P), we have
$$\begin{array}{rl} 4x_1 + 6x_2 &= 7 \\ 3x_1 + x_2 &= 3 \\ \hline \Rightarrow \quad 14x_1 &= 11 \end{array} \Rightarrow x_1 = \frac{11}{14}, x_2 = \frac{9}{14}.$$

Indeed, $z^* = 3x_1 + 3x_2 = 3\left(\frac{11+9}{14}\right) = \frac{30}{7} = w^*$, as expected.

AMPL model of (D)

```
var y1 >= 0;
var y2 >= 0;

maximize w: 7*y1 + 3*y2;

s.t.  x1:        4*y1 + 3*y2 <= 3;
s.t.  x2:        6*y1 +    y2 <= 3;
s.t.  x3:        3*y1 +    y2 <= 4;
```

AMPL session:

```
ampl: reset; model Pr6_PracFinal.txt; solve; display y1,y2;
Gurobi 10.0.0: optimal solution; objective 4.285714286
2 simplex iterations
y1 = 0.428571
y2 = 0.428571

ampl: display x1,x2,x3;
x1 = 0.785714
x2 = 0.642857
x3 = 0
```

$4.285714286 \rightarrow \frac{30}{7}$

$y_1, y_2 \rightarrow 3/7$

$x_1 \rightarrow 11/14$

$x_2 \rightarrow 9/14$

MATH 364 : Lecture 30 (12/05/2024)

Today: Practice final exam.

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3. Word selection IP:

Let $x_j = 1$ if word j is selected, and 0 otherwise, (0 or 1).

$j=1 \equiv$ AFT, $j=2 \equiv$ FAR, ..., $j=7 \equiv$ ZAP.

Let ℓ_i = sum of letter i scores, $i=1,2,3.$ (≥ 0)

Data: s_i = total score for word i.

s_1 = score(AFT) = 27, ..., s_7 = score(ZAP) = 43.

$$\max\ z = \sum_{i=1}^{7} s_i x_i \qquad \text{(total score)}$$

s.t.

$$\ell_1 = \underset{A}{x_1} + \underset{F}{6x_2} + \cdots + \underset{Z}{26x_7} \qquad \text{(letter 1 score)}$$

$$\ell_2 = \underset{F}{6x_1} + \underset{A}{x_2} + \cdots + \underset{A}{x_7} \qquad \text{(letter 2 score)}$$

$$\ell_3 = \underset{T}{20x_1} + \underset{R}{18x_2} + \cdots + \underset{P}{16x_7} \qquad \text{(letter 3 score)}$$

$$\sum_{i=1}^{7} x_i = 4 \qquad \text{(pick 4 words)}$$

$$x_2 \leq 1 - x_7 \qquad \text{(ZAP} \Rightarrow \text{no FAR)}$$

$$x_3 = x_4 \qquad \text{(JOE \& KEN, or neither)}$$

l_1, l_2, l_3 will all be integers. We want $l_1 < l_2 < l_3$.
Hence we can write

$l_1 \le l_2 - 1$ (letter 1 score < let. 2 score)

$l_2 \le l_3 - 1$ (letter 2 score < let. 3 score)

$x_j \in \{0,1\},\ j=1,\dots,7$ (Binary vars)

5. **if** $|2x + 5y| > 2$ **then** $|3x + 4y| \ge 5$.

$\equiv$ either $|2x+5y| \le 2$ or $|3x+4y| \ge 5$

$\equiv$ either $(2x+5y \le 2 \text{ AND } 2x+5y \ge -2)$ OR
$(3x+4y \ge 5 \text{ OR } 3x+4y \le -5)$

$\equiv$ either $(\overset{(1)}{2x+5y-2 \le 0}\ \underline{\text{AND}}\ \overset{(2)}{-2x-5y-2 \le 0})$ OR
$(\underset{(3)}{-3x-4y+5 \le 0}\ \text{OR}\ \underset{(4)}{3x+4y+5 \le 0})$

if $|x| \le 2$
then $-2 \le x \le 2$.

if $|x| \ge 3$
then $x \ge 3$ or
$x \le -3$

Let $t_i = 1$ if statement (i) holds; $i = 1,2,3,4$.
But (1) AND (2) is one option, so we use t_1 in place of t_2.

$2x+5y-2 \le M(1-t_1)$
$-2x-5y-2 \le M(1-t_1)$ } OR, you could use t_2 for (2), but write $t_1+t_2+t_3+t_4 \ge 1$ & $t_1 = t_2$.
$-3x-4y+5 \le M(1-t_3)$
$3x+4y+5 \le M(1-t_4)$
$t_1 + t_3 + t_4 \ge 1$
$t_i \in \{0,1\},\ i = 1,3,4$