MATH 364: Lecture 10 (09/19/2024)

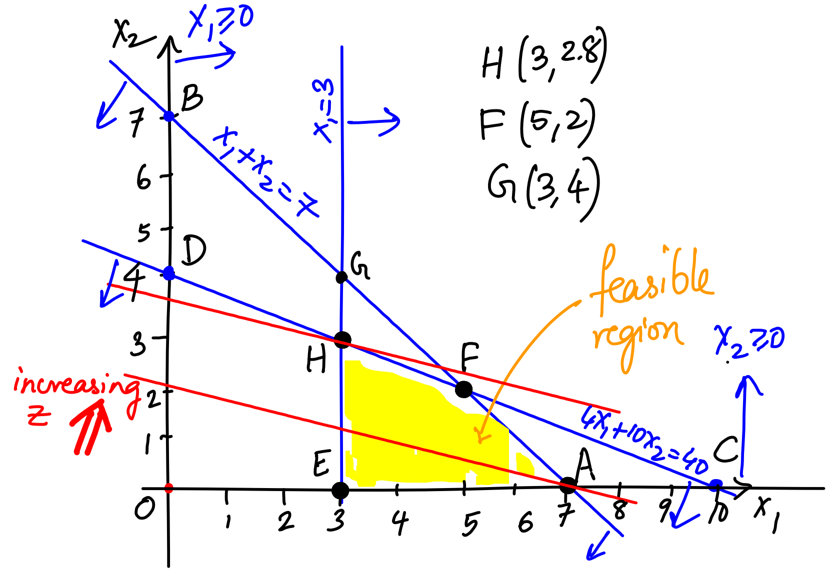
Today: * correspondence between bfs's & corner points
* simplex method

Farmer Jones LP

$$\max \; z = 30x_1 + 100x_2$$

s.t.

$$\begin{aligned} x_1 + x_2 + s_1 \quad &= 7 \\ 4x_1 + 10x_2 \quad + s_2 &= 40 \\ 10x_1 \quad - e_3 &= 30 \\ x_1, x_2, s_1, s_2, e_3 &\geq 0 \end{aligned}$$



H (3, 2.8)
F (5, 2)
G (3, 4)

We saw BV $= \{x_1, x_2, s_1\}$, NBV $= \{s_2, e_3\}$ gives the bfs $\equiv$ H (3, 2.8).

$$\bar{x} = \begin{array}{c} x_1 \\ x_2 \\ s_1 \\ s_2 \\ e_3 \end{array} \begin{bmatrix} 3 \\ 2.8 \\ 1.2 \\ 0 \\ 0 \end{bmatrix}.$$

For BV $= \{x_2, s_2, e_3\}$, NBV $= \{x_1, s_1\}$ gives the basic solution

$$\bar{x} = \begin{array}{c} x_1 \\ x_2 \\ s_1 \\ s_2 \\ e_3 \end{array} \begin{bmatrix} 0 \\ 7 \\ 0 \\ -30 \\ -30 \end{bmatrix}$$

$\rightarrow$ corresponds to B(0, 7), which is not feasible.

$$\left\{ \begin{aligned} x_2 \quad &= 7 \\ 10x_2 + s_2 &= 40 \\ -e_3 &= 30 \end{aligned} \right\} \Longrightarrow x_2 = 7,\ e_3 = -30,\ s_2 = -30.$$

We could identify the bfs corresponding to each corner point directly from the picture!

Let's consider $F(5,2)$.

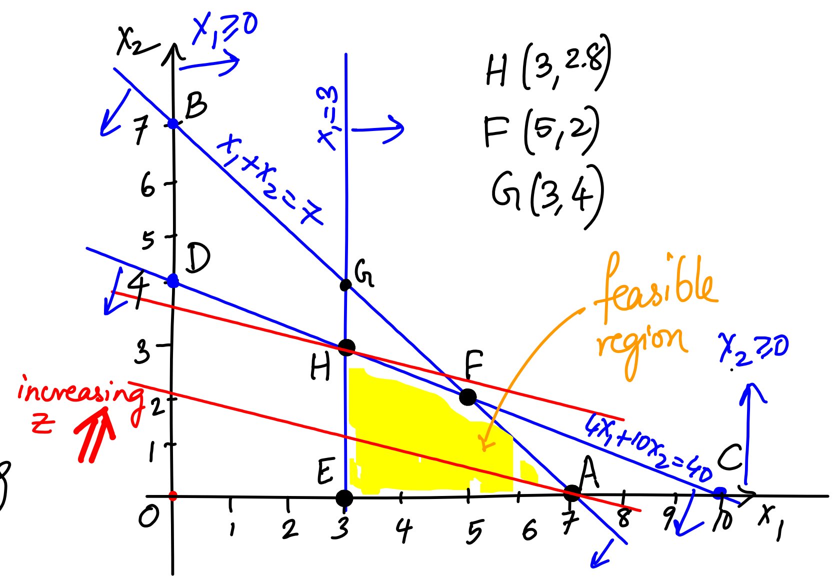
$x_1=5,\ x_2=2$ are in BV.

At F, the (land_avai) and (labor_hrs) constraints are binding (i.e., satisfied as equalities). Hence $s_1=0$ and $s_2=0$. But the (min_corn) constraint is non-binding at F, hence $e_3>0$. Hence

$BV=\{x_1,x_2,e_3\}$, $NBV=\{s_1,s_2\}$,

and the bfs is
$$\bar{X}=\begin{array}{c}x_1\\x_2\\s_1\\s_2\\e_3\end{array}\begin{bmatrix}5\\2\\0\\0\\20\end{bmatrix}.$$

$10(5)-30$



We can present the correspondence between corner points and bfs's in a table as shown below.

Correspondence between bfs's and corner points

Corner point	BV	NBV	BFS $[x_1\ x_2\ s_1\ s_2\ e_3]$
A (7, 0)	x_1, e_3, s_2	x_2, s_1	[7 0 0 12 40]
E (3, 0)	x_1, s_1, s_2	x_2, e_3	[3 0 4 28 0]
H (3, 2.8)	x_1, x_2, s_1	s_2, e_3	[3 2.8 1.2 0 0]
F (5, 2)	x_1, x_2, e_3	s_1, s_2	[5 2 0 0 20]

Let's summarize a bit. We have seen the following results.

* If the LP in standard form has an optimal solution, there must be a corner point that is optimal.

* Corner points $\Longleftrightarrow$ bfs

Hence we get the following result.

Theorem If an LP in standard form has an optimal solution, then it has an optimal bfs.

The simplex method explores the corner points, or bfss. The idea is to start at one bfs, and move to a neighboring bfs (or corner point) at which the objective function is better. This procedure is equivalent to sliding the z-line in 2D.

While the idea of a neighboring corner point is straightforward to imagine in 2D, we switch to an algebraic view in higher dimensions using the correspondence given above.

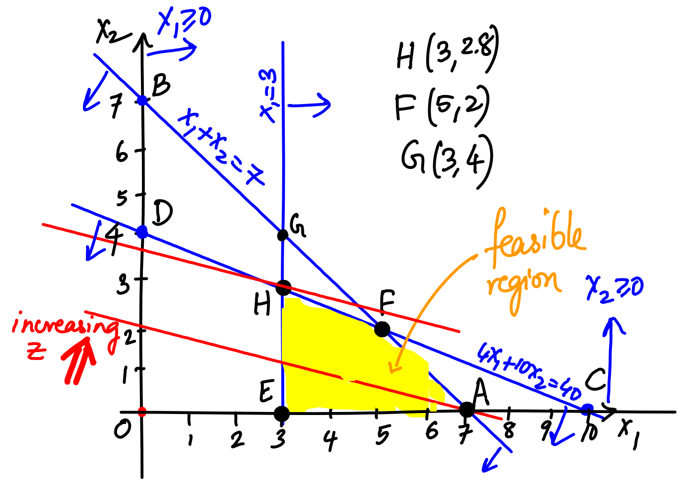
Simplex Method

* Start at a bfs $\equiv$ corner point.
* If not optimal, move to a "nearby" (adjacent) bfs so that the z-value improves
* If no such "better" adjacent bfs exists, the current corner point $\equiv$ bfs is optimal.

Adjacent bfs

Def For an LP in standard form with n variables and m constraints ($m \le n$, rank$(A) = m$), two bfs's are said to be **adjacent** if they have $(m-1)$ common basic variables.

Corner point	BV	NBV	BFS $[x_1\ x_2\ s_1\ s_2\ e_3]$
A (7,0)	x_1, e_3, s_2	x_2, s_1	[7 0 0 12 40]
E (3,0)	x_1, s_1, s_2	x_2, e_3	[3 0 4 28 0]
H (3, 2.8)	$\boxed{x_1}, \boxed{x_2}, s_1$	s_2, e_3	[3 2.8 1.2 0 0]
F (5,2)	$\boxed{x_1}, \boxed{x_2}, e_3$	s_1, s_2	[5 2 0 0 20]



For instance we could start at A (7,0), where $z = 210$, and move to the adjacent bfs F(5,2), where $z = 350$. Notice that the other adjacent bfs to A is E(3,0). But $z = 90$ at E, and hence we do not move to E.

Repeating the same procedure, we move from F to the adjacent bfs H(3, 2.8), where $z = 370$. From H, both adjacent corner points (F and E) have smaller z-values, and hence we can conclude that H is an optimal solution.

We could, alternatively start at E and move to H directly.

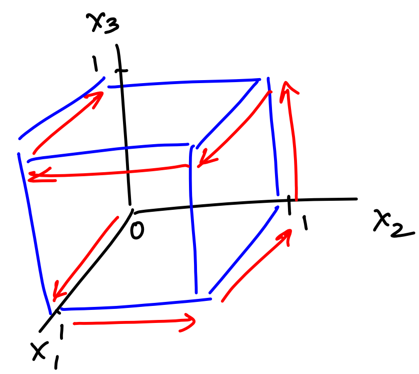
How many bfs's are there?

Could solve LP by evaluating z at every bfs (corner point).
What is the max # bfs's an LP can have?

For each bfs we need to choose m basic variables out of n variables.
Hence there are $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ choices, which could be quite large!

But not all of these choices might lead to a bfs

On the one hand, there are artificially constructed LP instances for which every version of the simplex algorithm would have to inspect all of $\binom{n}{m}$ bfss (all of them being feasible, of course).



2^n vertices in general!

But on the other hand most LPs arising out of applications tend not to exhibit such structure, and the simplex method is usually very fast in solving most LPs.

Simplex Method

Simplex Algorithm for maximization LPs

Step 1 Convert LP to standard form.

Step 2 Obtain a bfs from the standard form.

Step 3 Find if current bfs is optimal.
If YES, STOP.

Step 4 If current bfs is not optimal, find which non-basic variable should become basic, and which basic variable should become non-basic in order to move to an adjacent bfs with a higher objective function value.
we are solving a max LP.

Step 5 Use EROs to obtain the adjacent bfs.
Return to Step 3.

We specify more details for each step as we illustrate the simplex algorithm on an example. We start with an LP where all constraints are '$\leq$'. Step 2 becomes easy in this case. We will discuss how to deal with '$\geq$' and '$=$' constraints later on. We will assume also that all variables are non-negative for now.

Solve the following LP using the simplex method

$$\max \quad z = 2x_1 + 3x_2$$
$$\text{s.t.} \quad x_1 + 2x_2 \le 6 \quad s_1 \ge 0$$
$$2x_1 + x_2 \le 8 \quad s_2 \ge 0$$
$$x_1, x_2 \ge 0$$

Step 1

$$\max \quad z = 2x_1 + 3x_2$$
$$\text{s.t.} \quad x_1 + 2x_2 + s_1 = 6$$
$$2x_1 + x_2 + s_2 = 8$$
$$x_1, x_2, s_1, s_2 \ge 0$$

Step 2 We first write the LP in a more organized manner:

$$\left.\begin{array}{llllllll} 0. & z & -2x_1 & -3x_2 & & & = & 0 \\ 1. & & x_1 & +2x_2 & +s_1 & & = & 6 \\ 2 & & 2x_1 & +x_2 & & +s_2 & = & 8 \end{array}\right\} \text{canonical form}$$

Def An LP is written in **canonical form** if each row including Row-0 has a variable (including z) with coefficient 1 in that row and zero in every other row.

Here, $\{z, s_1, s_2\}$ is the set of canonical variables.

We can choose z and the remaining m canonical variables in the starting bfs. If the rhs (b_i for the i^{th} constraint) values are all ≥ 0, we can read off the bfs from the canonical form.

Here, we set $x_1 = x_2 = 0$, and get $s_1 = 6$, $s_2 = 8$, and $z = 0$.