MATH 364: Lecture 11 (09/24/2024)

Today: * simplex for max LP
* tableau simplex

Simplex Algorithm for maximization LPs

Step 1 Convert LP to standard form.

Step 2 Obtain a bfs from the standard form.

Step 3 Find if current bfs is optimal.
If YES, STOP.

Step 4 If current bfs is not optimal, find which non-basic variable should become basic, and which basic variable should become non-basic in order to move to an adjacent bfs with a higher objective function value.

Step 5 Use EROs to obtain the adjacent bfs.
Return to Step 3.

Recall the steps of the simplex method for max-LP

We will continue with the example from Lecture 10:

$$\begin{aligned} \max\ & z = 2x_1 + 3x_2 \\ \text{s.t. } & x_1 + 2x_2 \le 6 \quad s_1 \ge 0 \\ & 2x_1 + x_2 \le 8 \quad s_2 \ge 0 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Step 1

$$\begin{aligned} \max\ & z = 2x_1 + 3x_2 \\ \text{s.t. } & x_1 + 2x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned}$$

Step 2

$$\left.\begin{aligned} &0.\quad z - 2x_1 - 3x_2 = 0 \\ &1.\quad x_1 + 2x_2 + s_1 = 6 \\ &2.\quad 2x_1 + x_2 + s_2 = 8 \end{aligned}\right\}$$

Can read off the bfs from the LP in canonical form: Here, $s_1 = 6$, $s_2 = 8$ is the bfs, giving $z = 0$.

$BV = \{z, s_1, s_2\}$, $NBV = \{x_1, x_2\}$.

Step 3 Check if current bfs is optimal.

bfs is optimal if we cannot improve the z-value by increasing the value of any non-basic variable (from 0).

Here, $z = 2x_1 + 3x_2 = 0$ now (right now $x_1 = x_2 = 0$).

If $x_1 = 1$, z becomes 2
If $x_2 = 1$, z becomes 3

So, current bfs is not optimal.
We will see in Step 4 which of these two vars we will increase, and by how much.

Step 4 Want to move to an adjacent bfs such that the z-value increases.

We want to consider increasing one non-basic variable from 0 to a nonzero value, as we want to move to an adjacent bfs, which shares all but one basic variable with the current bfs.

We could increase either x_1 or x_2 to improve z. By default, we pick the non-basic variable that has the largest rate of increase – here, it's x_2. Hence x_2 is the **entering variable.**

x_2 enters

$$
\begin{array}{llllll}
0. & z & -2x_1 & -3x_2 & & = 0 \\
1. & & x_1 & +2x_2 & +s_1 & = 6 \\
2. & & 2x_1 & +x_2 & +s_2 & = 8
\end{array}
$$

If we keep increasing x_2 without limit, we might make one of the currently basic variable negative, i.e., infeasible.

$$\text{Row 1: } \quad 2x_2 + s_1 = 6 \quad \Rightarrow \quad s_1 = 6 - 2x_2$$

$$\text{Row 2: } \quad x_2 + s_2 = 8 \quad \Rightarrow \quad s_2 = 8 - x_2$$

To keep $s_1 \geq 0$, we cannot increase x_2 beyond $6/2 = 3$, i.e., $x_2 \leq 3$

Similarly to keep $s_2 \geq 0$, $x_2 \leq 8$.

Choosing the smaller of the two limits, we get $x_2 \leq 3$.

On the other hand, if the dependence of s_1 on x_2 were specified as $s_1 = 6 + 2x_2$, for instance, there will be no limit placed on the value of x_2 in this case. Similarly if the value of s_2 did not depend on x_2, e.g., $s_2 = 8$, we would not get an upper bound on x_2.

We formalize these observations into the minimum ratio test (min ratio test, in short) for picking which variable leaves the basis.

Minimum Ratio Test (min-ratio test)

For each constraint row that has a **positive coefficient** for the entering variable, compute the ratio

$$\frac{\text{right-hand side of row}}{\text{coefficient of entering var in row}}.$$

The smallest among all these ratios is the largest value the entering variable can take.

Here: Row 1: $6/2 = 3$; Row 2: $8/1 = 8$ $\}$ min-ratio $= 3$.

The variable that is basic (or canonical) in the row that is the winner of the min-ratio test is the **leaving variable**.

Here, s_1 leaves the basis.

Step 5 Make entering variable basic (or canonical) in the row that won the min-ratio test using EROs.

Here, make x_2 basic in Row 1, i.e., make coefficient of x_2 in Row 1 $= 1$, and 0 in other rows (including Row-0).

$$
\begin{array}{lllll}
0. & z - 2x_1 - 3x_2 & = 0 & & R_1 \times \left(\tfrac{1}{2}\right), \text{ then} \\
1. & x_1 + 2x_2 + s_1 & = 6 & 6/2 = 3 & R_0 + 3R_1,\ R_2 - R_1 \\
2 & 2x_1 + x_2 + s_2 & = 8 & 8/1 = 8 & \\
\hline
 & z - \tfrac{1}{2}x_1 + \tfrac{3}{2}s_1 & = 9 & & BV = \{z, x_2, s_2\} \\
 & \tfrac{1}{2}x_1 + x_2 + \tfrac{1}{2}s_1 & = 3 & 3/(1/2) = 6 & x_1 \text{ enters} \\
 & \tfrac{3}{2}x_1 - \tfrac{1}{2}s_1 + s_2 & = 5 & 5/(3/2) = 10/3 & R_2(2/3),\ R_0 + \tfrac{1}{2}R_2, \\
\hline
 & z + \tfrac{4}{3}s_1 + \tfrac{1}{3}s_2 & = 32/3 & & R_1 - \tfrac{1}{2}R_2 \\
 & x_2 + \tfrac{2}{3}s_1 - \tfrac{1}{3}s_2 & = 4/3 & & \\
 & x_1 - \tfrac{1}{3}s_1 + \tfrac{2}{3}s_2 & = 10/3 & & \\
\hline
\end{array}
$$

we perform all steps of the next iteration here

$BV = \{z, x_1, x_2\}$ is optimal, as $z = 32/3 - \tfrac{4}{3}s_1 - \tfrac{1}{3}s_2$, and increasing either s_1 or s_2 from 0 will decrease z.

We performed two iterations of the simplex method above.

Consider the following LP:

$$\max\ z = 2x_1 - x_2 + x_3$$

s.t.
$$3x_1 + x_2 + x_3 \le 60 \quad s_1$$
$$x_1 - x_2 + 2x_3 \le 10 \quad s_2$$
$$x_1 + x_2 - x_3 \le 20 \quad s_3$$
$$x_1, x_2, x_3 \ge 0$$

We can represent all the numbers in a compact table format, called the simplex tableau (pronounced "tablo"). All calculations are also efficiently represented in this format. This version of the simplex method is called the **tableau simplex method**.

Each tableau corresponds to a bfs, assuming it is constructed correctly. In fact, we could directly go to the starting tableau from the given LP.

$$\max\ z = 2x_1 - x_2 + x_3$$

s.t.
$$3x_1 + x_2 + x_3 \le 60 \quad s_1$$
$$x_1 - x_2 + 2x_3 \le 10 \quad s_2$$
$$x_1 + x_2 - x_3 \le 20 \quad s_3$$
$$x_1, x_2, x_3 \ge 0$$

starting tableau

$z - 2x_1 + x_2 - x_3 = 0$

x_1 enters, s_2 leaves

pivots

z	x_1	x_2	x_3	s_1	s_2	s_3	rhs
1	−2	1	−1	0	0	0	0
0	3	1	1	1	0	0	60
0	1	−1	2	0	1	0	10
0	1	1	−1	0	0	1	20
1	0	−1	3	0	2	0	20
0	0	4	−5	1	−3	0	30
0	1	−1	2	0	1	0	10
0	0	2	−3	0	−1	1	10
1	0	0	3/2	0	3/2	1/2	25
0	0	0	1	1	−1	−2	10
0	1	0	1/2	0	1/2	1/2	15
0	0	1	−3/2	0	−1/2	1/2	5

$R_0 + 2R_2$
$R_1 - 3R_2$
$R_3 - R_2$

$R_3(\frac{1}{2})$, then
$R_0 + R_3$, $R_1 - 4R_3$,
$R_2 + R_3$

all #'s (under variables) in Row-0 are ≥ 0
$\Longrightarrow$ tableau (i.e., bfs) is optimal!

optimal z-value is given as z^*.

The optimal solution is $x_1 = 15$, $x_2 = 5$, $s_1 = 10$, and $z^* = 25$.

Current bfs is optimal (for a max LP) if the numbers for each variable in Row-0 of the simplex tableau are nonnegative.

Let us recall the idea of the min ratio test, explaining it on the first tableau. Here, $BV = \{s_1, s_2, s_3\}$, $NBV = \{x_1, x_2, x_3\}$. Increasing x_1 or x_3 (from zero) will increase the z-value. We pick x_1, as the rate of increase is higher. Thus, x_1 is the entering variable.

Our goal is to move to an adjacent bfs at which the z-value is better (larger for a max LP). To move to an adjacent bfs, we exchange one basic variable with a current nonbasic variable. Here, we are going to include x_1 in the basis, and remove one of the current basic variables from the BV set. The min-ratio test helps us to identify the leaving variable.

The 3 constraint equations in the first tableau read as follows.

$$\left.\begin{array}{l} 3x_1 + s_1 \qquad\quad = 60 \\ x_1 \qquad + s_2 \quad = 10 \\ x_1 \qquad\qquad + s_3 = 20 \end{array}\right\} \Rightarrow \begin{array}{l} s_1 = 60 - 3x_1 \\ s_2 = 10 - x_1 \\ s_3 = 20 - x_1 \end{array}$$

We need to keep $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$ for feasibility. Hence we get $60 - 3x_1 \geq 0$, $10 - x_1 \geq 0$, $20 - x_1 \geq 0$, or equivalently,

$x_1 \leq 60/3$, $x_1 \leq 10$, $x_1 \leq 20$, which all hold when $x_1 \leq 10$.

When $x_1 > 10$, s_2 becomes negative, i.e., we are no longer feasible. So $x_1 = \frac{10}{1} = 10$ is the winner of the min ratio test, and since this ratio comes from Row 2, in which s_2 is canonical at present, the entering variable x_1 replaces s_2 from BV set (i.e., s_2 leaves the basis).

Notice that if we had $s_2 = 10 + x_1$ (instead of $-$), then increasing x_1 would not affect the nonnegativity of s_2. This is the reason why we do not consider rows for the min ratio test that have negative (or zero) coefficients for the entering variable.