MATH 364: Lecture 12 (09/27/2024)

Today:
* simplex for min LPs
* alternative optimal solutions in simplex method
* unbounded LPs
* big-M simplex method

Simplex method for min LPs

The criteria to decide entering variable and optimality of the bfs are opposite to those used in a max LP.

* Current bfs is optimal if all numbers in Row-0 for variables are ≤ 0 (non-positive).
* Nonbasic variable with the largest positive number in Row-0 enters (default rule for entering variable).
* min-ratio test: same as in max LP.

$$\min \; z = 4x_1 - x_2$$
s.t.
$$2x_1 + x_2 \leq 8 \quad s_1$$
$$x_2 \leq 5 \quad s_2$$
$$x_1 - x_2 \leq 4 \quad s_3$$
$$x_1, x_2 \geq 0$$

BV	z	x_1	x_2	s_1	s_2	s_3	rhs
	1	−4	1	0	0	0	0
s_1	0	2	1	1	0	0	8
s_2	0	0	1	0	1	0	5
s_3	0	1	−1	0	0	1	4
	1	−4	0	0	−1	0	−5
s_1	0	2	0	1	−1	0	3
x_2	0	0	1	0	1	0	5
s_3	0	1	0	0	1	1	9

Current tableau is optimal, as all #'s in Row-0 under variables are non-positive. Optimal solution is

$x_2 = 5$, $s_1 = 3$, $s_3 = 9$, and $z^* = -5$.

Another approach for min-LPs

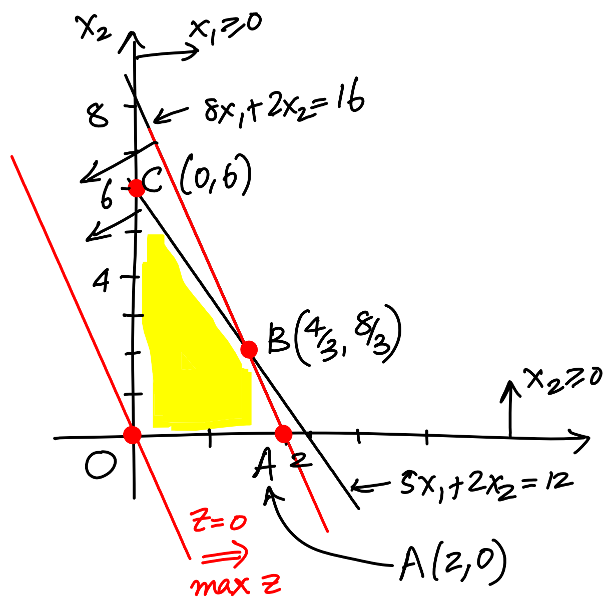
Instead of solving $\left\{\begin{array}{ll}\min & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}\le\bar{b} \\ & \bar{x}\geq\bar{0}\end{array}\right\}$, solve $\left\{\begin{array}{ll}\max & -\bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}\le\bar{b} \\ & \bar{x}\geq\bar{0}\end{array}\right\}$

using the criteria for max-LP. Set z^*_{min} as $-z^*_{max}$, where z^*_{max} is the z^* for the max-LP. The optimal $\bar{x}$ remains same.

Alternative Optimal Solutions

Recall LP from Lecture 5:

$$\begin{array}{ll}\max & z = 4x_1+x_2 \\ \text{s.t.} & 8x_1+2x_2\le 16 \\ & 5x_1+2x_2\le 12 \\ & x_1, x_2 \geq 0\end{array}$$



Both A and B, as well as any point on $\overline{AB}$ are optimal solutions.

In 3D, we could have 3 or more vertices which are all optimal at the same time, and the "side" defined by all of them constitute the (infinite number of) alternative optimal solutions (similar to segment $\overline{AB}$ here)

We will have more than one optimal tableau, corresponding to each optimal bfs.

$\max\ z = x_1 + x_2$

s.t. $x_1 + x_2 + x_3 \leq 1$

$x_1 \qquad + 2x_3 \leq 1$

$x_1, x_2, x_2 \geq 0$

break ties arbitrarily

BV	z	x_1	x_2	x_3	s_1	s_2	rhs	
	1	−1	−1	0	0	0	0	
s_1	0	1	1	1	1	0	1	
s_2	0	1	0	2	0	1	1	
	1	0	0	1	1	0	1	optimal tableau
x_1	0	1	1	1	1	0	1	
s_2	0	0	−1	1	−1	1	0	
	1	0	0	1	1	0	1	optimal tableau
x_2	0	1	1	1	1	0	1	
s_2	0	1	0	2	0	1	1	
	1	0	0	1	1	0	1	optimal tableau
x_2	0	0	1	−1	1	−1	0	
x_1	0	1	0	2	0	1	1	

In the last tableau, s_2 has coefficient zero in Row-0, and could enter the basis. But we'll get back the previous optimal tableau.

Criterion:

If the coefficient of a non-basic variable in Row-0 of an optimal tableau is zero, there exist alternative optimal solutions. If we can pivot this variable into the basis, then there are alternative optimal bfs's.

There are 3 optimal bfs's here, corresponding to

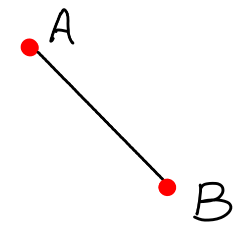
$x_1 = 1,\ x_2 = 0$, $\quad x_2 = 1,\ s_2 = 1$, and $\quad x_1 = 1,\ s_2 = 0$.

But in terms of $\{x_1, x_2, x_3\}$, these 3 bfs's correspond to two optimal solutions

original variables

$$A = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Also, any point on the line segment $\overline{AB}$ is optimal, i.e., any $\bar{x} = \alpha A + (1-\alpha) B$, $0 \leq \alpha \leq 1$ is optimal.



$$\bar{x} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\begin{bmatrix} \alpha \\ 1-\alpha \\ 0 \end{bmatrix}, \quad 0 \leq \alpha \leq 1.$$

this expression is analogous to the parametric vector form of solutions to $A\bar{x} = b$, when there are free variables.

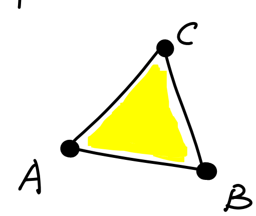
For instance, $\alpha = 0.5$, we get the mid point of $\overline{AB}$.

Indeed, $z = x_1 + x_2 = \alpha + 1 - \alpha = 1 = z^*$ for any such α.

With 3 different optimal vertices A, B, C, all optimal solutions can be written as

$$\bar{x} = \alpha_A A + \alpha_B B + \alpha_C C, \quad 0 \leq \alpha_A, \alpha_B, \alpha_C \leq 1$$
$$\alpha_A + \alpha_B + \alpha_C = 1$$

e.g., $\alpha_A = \frac{1}{2}, \alpha_C = \frac{1}{2}, \alpha_B = 0$, gives the mid point of '$\overline{AC}$'.

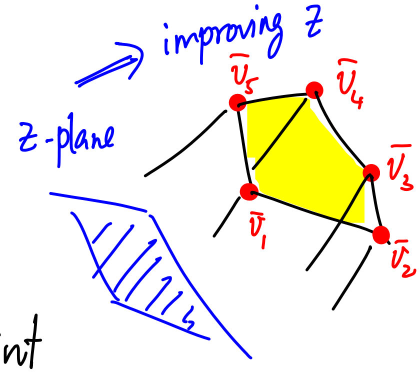


$\bar{x}$ here is a **convex combination** of A, B, C.

A linear combination is $\bar{x} = \alpha_A A + \alpha_B B + \alpha_C C$, for $\alpha_A, \alpha_B, \alpha_C \in \mathbb{R}$.

Thus, a convex combination is a special linear combination.

Idea in 3D (and higher dimensions): Example



The z-plane hits flush against an entire face, here shown with five corner points $\bar{v}_j$, $j=1-5$, for instance

Each corner point $\bar{v}_j$ is optimal, and so is any point in the shaded region. Any point in the pentagon is a convex combination of the $\bar{v}_j$s. → more generally, there could be many $\bar{v}_j$'s (not just 5).

Def A **convex combination** of $\bar{v}_1, \ldots, \bar{v}_n$ is

$$\bar{x} = \sum_{j=1}^{n} \alpha_j \bar{v}_j, \quad 0 \le \alpha_j \le 1, \quad \sum_{j=1}^{n} \alpha_j = 1.$$

For instance, when $\alpha_2 = 1$, $\alpha_j = 0$ for $j=1,3,4,5$, $\bar{x} = \bar{v}_2$. Similarly, when $\alpha_3 = \alpha_5 = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \alpha_4 = 0$, we get $\bar{x} = \frac{1}{2}(\bar{v}_3 + \bar{v}_5)$, which is the midpoint of the line segment connecting $\bar{v}_3$ and $\bar{v}_5$. And when $\alpha_j = \frac{1}{5}$ for all j, $\bar{x}$ is the "centroid" (or average) of all the corner points.

Unbounded LPs

Recall that in 2D, when you could slide the z-line without limits while improving z and remaining feasible, the LP is unbounded.

max $z = 2x_2$

s.t.
$$x_1 - x_2 \le 4$$
$$-x_1 + x_2 \le 1$$
$$x_1, x_2 \ge 0$$

BV	z	x_1	x_2	s_1	s_2	rhs
	1	0	−2	0	0	0
s_1	0	1	−1	1	0	4
s_2	0	−1	1	0	1	1
	1	−2	0	0	2	2
s_1	0	0	0	1	1	5
x_2	0	−1	1	0	1	1

We do not have any candidates for the min-ratio test in the second tableau $\Rightarrow$ LP is unbounded. x_1 could enter the basis and improve the z-value, but there is no limit on how much the increase can be.

The equations (in Rows 1 & 2) are

$$\left.\begin{array}{l} s_1 = 5 \\ -x_1 + x_2 = 1 \Rightarrow x_2 = 1 + x_1 \end{array}\right\}$$ as x_1 increases, both s_1 and x_2 stay > 0.

Thus we could keep increasing x_1, and hence improving z, without ever ever encountering infeasibility. Hence the LP is unbounded!

Criterion: The tableau has a non-basic variable that could enter and improve the value of z, but there are no candidates for min-ratio.

the coefficient cannot be zero

So far, the LPs we have looked at are all of the form

$$\left\{\begin{array}{l} \max/\min\ \bar{c}^T\bar{x} \\ A\bar{x} \le \bar{b} \\ \bar{x} \geq \bar{0} \end{array}\right\}$$

where $\bar{b} \geq \bar{0}$. $\bar{x}=\bar{0}$ is always feasible here.

So, we do not get infeasible LPs.

To consider infeasible LPs, we introduce a general tableau simplex method that could handle $\geq$ and $=$ constraints.

The big-M Method of Tableau Simplex

Can handle $\geq$ or $=$ constraints

IDEA
* add artificial variables in order to obtain a starting bfs.
* modify objective function so as to force the artificial variables to zero in the optimal solution.

$$\begin{array}{ll} \min & z = 2x_1+3x_2 \\ \text{s.t.} & 2x_1+x_2 \geq 4 \quad \text{———} \ (1) \\ & x_1-x_2 \geq -1 \quad \text{———} \ (2) \\ & x_1, x_2 \geq 0 \end{array}$$

Step 1 Modify any constraints so that all rhs values are nonnegative.

Recall that we can read off the bfs from the tableau — assuming all rhs values are ≥ 0. Else, feasibility is violated.

If the rhs value of a constraint is negative, scale it by -1. The sense of the inequality is reversed here.

$$(2) \times -1 \Rightarrow -1(x_1-x_2 \geq -1) \qquad -x_1+x_2 \le 1 \quad \text{———} \ (2')$$

For instance, consider $-3 \geq -5$. Multiplying this inequality by -1 indeed reverses the sense of the inequality: $-(-3 \geq -5) \Rightarrow 3 \le 5$.

One advantage of using slack variables is that we can choose the obvious starting bfs by picking the slack variables in the BV. But for '$\geq$' constraints, we subtract excess variables, which are not canonical. Similarly, we do not have obvious canonical variables for '$=$' constraints. Hence, we add artificial variables for such constraints.

Step 2 Add an artificial variable a_i to constraint i if it is a $\geq$ or $=$ constraint, and add $a_i \geq 0$.

$$(1) \Rightarrow \quad 2x_1 + x_2 + a_1 \geq 4 \quad \text{——} (1')$$

Step 3 For max-LP, add $-Ma_i$ to the objective function (z); and for min-LP, add $+Ma_i$ to z, where M is a large positive number.

$$\begin{aligned} \min \quad & z = 2x_1 + 3x_2 + Ma_1 & & \\ \text{s.t.} \quad & 2x_1 + x_2 + a_1 \geq 4 & & \text{——} (1') \\ & -x_1 + x_2 \leq 1 & & \text{——} (2') \\ & x_1, x_2, a_1 \geq 0 & & \end{aligned}$$

this term forces a_1 to zero in any optimal solution, assuming the LP is not infeasible. With the M coefficient, as long as $a_1 > 0$, z is very huge due to the Ma_1 term, however small $a_1 > 0$ is.

M acts like ∞, but we can "handle" it!

So $\quad 3M + 1 > 2M + 123456$

$$-2M + 10 < -M - 2500000$$

Step 4 Convert all inequalities to standard form (using slack/excess vars).

$$\begin{aligned}
\min\ & z = 2x_1 + 3x_2 + Ma_1 \\
\text{s.t.}\ & 2x_1 + x_2 + a_1 - e_1 = 4 \quad \text{——} \ (1') \\
& -x_1 + x_2 + s_2 = 1 \quad \text{——} \ (2') \\
& x_1, x_2, a_1, e_1, s_2 \geq 0
\end{aligned}$$

We will describe the remaining steps in the next lecture...