MATH 364: Lecture 13 (10/01/2024)

Next Tuesday (Oct 8): Midterm (in-class; practice midterm is posted)
Topics: Everything before big-M method

Today: * infeasibility in tableau simplex
* urs vars

We finish describing the steps of the big-M simplex method.
Recall steps 1–4 from Lecture 12...

Step 5 Convert LP to canonical form by converting the coefficients of artificial variables a_i in Row-0 to zero (using EROs). The initial bfs will then have all slack (s_i) and all artificial variables (a_i's). Solve the resulting LP tableau using regular simplex method.

BV	z	x_1	x_2	e_1	s_2	a_1	rhs
	1	-2	-3	0	0	$-M$	0
	0	2	1	-1	0	1	4
	0	-1	1	0	1	0	1
	1	$2M-2$	$M-3$	$-M$	0	0	$4M$
a_1	0	2	1	-1	0	1	4
s_2	0	-1	1	0	1	0	1
	1	0	-2	-1	0	$-(M-1)$	4
x_1	0	1	$1/2$	$-1/2$	0	$1/2$	2
s_2	0	0	$3/2$	$-1/2$	1	$1/2$	3

$R_0 + MR_1$

$R_0 - (2M-2)R_1$ ← new R_1

$\cancel{M}-3-(2\cancel{M}-2)\frac{1}{2}$

$-\cancel{M}+(2\cancel{M}-2)\frac{1}{2}$

$4\cancel{M}-(2\cancel{M}-2)2$

Optimal solution: $x_1=2$, $s_2=3$, $z^*=4$.

If any a_i is >0 in the optimal tableau (i.e., it is basic), the original LP is infeasible.

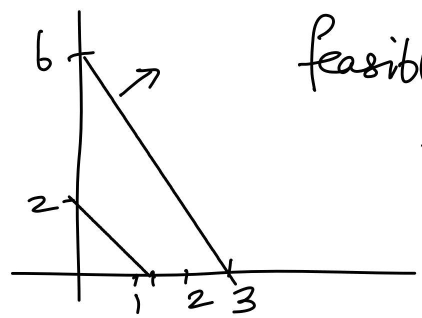
Detecting infeasible LPs

Recall that if all constraints are $\le$, all rhs values (b_i's) are ≥ 0, then $\bar{x} = \bar{0}$ is feasible. But in more general settings, we can detect infeasible LPs using the big-M simplex method.

Criterion

If an artificial var is basic in the optimal tableau (i.e., is > 0), then the original LP is infeasible.

$$\min\ z = 3x_1 + Ma_1 + Ma_2$$
$$\text{s.t.}\quad 2x_1 + x_2 + a_1 \ge 6 \quad e_1$$
$$3x_1 + 2x_2 + a_2 = 4$$
$$x_1, x_2, a_1, a_2, e_1 \ge 0$$



feasible region is empty

BV	z	x_1	x_2	e_1	a_1	a_2	rhs
	1	-3	0	0	$-M$	$-M$	0
	0	2	1	-1	1	0	6
	0	3	2	0	0	1	4
	1	$5M-3$	$3M$	$-M$	0	0	$10M$
a_1	0	2	1	-1	1	0	6
a_2	0	3	2	0	0	1	4
	1	0	$-M/3+2$	$-M$	0	$-\frac{5}{3}M+1$	$\frac{10}{3}M+4$
a_1	0	0	$-1/3$	-1	1	$-2/3$	$10/3$
x_1	0	1	$2/3$	0	0	$1/3$	$4/3$

$R_0 + MR_1 + MR_2$

optimal!

$R_0 - (5M-3)R_2 \rightarrow$ new R_2

$3M - (5M-3)\frac{2}{3}$

$10M - (5M-3)4/3$

Since $a_1 = 10/3$ in the optimal tableau, the original LP is infeasible.

Unrestricted in Sign (Urs) Variables

* If x_i is urs, replace x_i by $x_i^+ - x_i^-$ in all constraints and in the objective function, and add $x_i^+, x_i^- \geq 0$.

IDEA: $x_i = x_i^+ - x_i^-$ $(x_i^+, x_i^- \geq 0)$

If $x_i = 5$, $x_i^+ = 5$, $x_i^- = 0$ works, and

if $x_i = -3$, $x_i^+ = 0$, $x_i^- = 3$ works.

$x_i^+ = 8$, $x_i^- = 3$ works too, but we will show only one of x_i^+, x_i^- can be basic in the optimal tableau.

$$\max\ z = 2x_1 + \underbrace{x_2}_{x_2^+ - x_2^-}$$
$$\text{s.t.}\quad 3x_1 + x_2 \leq 6$$
$$x_1 + x_2 \leq 4$$
$$x_1 \geq 0,\ x_2 \text{ urs}$$

$$\max\ z = 2x_1 + x_2^+ - x_2^-$$
$$\text{s.t.}\quad 3x_1 + x_2^+ - x_2^- \leq 6 \quad s_1$$
$$x_1 + x_2^+ - x_2^- \leq 4 \quad s_2$$
$$x_1, x_2^+, x_2^- \geq 0$$

BV	z	x_1	x_2^+	x_2^-	s_1	s_2	rhs
	1	−2	−1	1	0	0	0
s_1	0	3	1	−1	1	0	6
s_2	0	1	1	−1	0	1	4
	1	0	−1/3	1/3	2/3	0	4
x_1	0	1	1/3	−1/3	1/3	0	2
s_2	0	0	2/3	−2/3	−1/3	1	2
	1	0	0	0	1/2	1/2	5
x_1	0	1	0	0	1/2	−1/2	1
x_2^+	0	0	1	−1	−1/2	3/2	3

optimal solution $x_1 = 1$, $x_2 = x_2^+ - x_2^- = 3 - 0 = 3$, $z^* = 5$.

Note that the columns of x_2^+ and x_2^- are -1 multiples of each other. Hence both cannot be basic in a tableau, and we get x_2 modeled correctly.

If $x_i \le 0$ to start with, replace x_i by $-x_i'$ everywhere, and add $x_i' \ge 0$. In the end set $x_i = -x_i'$ (in optimal solution).
instead of it being urs

Putting it all together

$$\begin{aligned} \min\ & Z = 2x_1 - 3x_2 \\ \text{s.t.}\ & x_1 + 3x_2 \le 9 \\ & 2x_1 + 5x_2 \ge -6 \\ & x_2 \ge 1 \\ & x_1 \text{ urs},\ x_2 \ge 0 \end{aligned}$$

Solve this LP using the big-M method.

We outline the steps of big-M simplex method.

Step 1 Scale any constraint with <0 rhs

$$-(2x_1 + 5x_2 \ge -6) \Rightarrow \quad -2x_1 - 5x_2 \le 6$$

Steps 2&3

Add artificial var a_i to constraint i if it is $\ge$ or $=$; and add $a_i \ge 0$.
add $\pm Ma_i$ to z (obj. fn) ($+Ma_i$ for min LP).

$$\begin{aligned} \min\ & Z = 2x_1 - 3x_2 + Ma_3 \\ \text{s.t.}\ & x_1 + 3x_2 \le 9 \quad s_1 \\ & -2x_1 - 5x_2 \le 6 \quad s_2 \\ & x_2 + a_3 \ge 1 \quad e_3 \\ & x_1 \text{ urs},\ x_2 \ge 0,\ a_3 \ge 0 \end{aligned}$$

Step 4 Replace urs variable x_i by $x_i^+ - x_i^-$, add $x_i^+, x_i^- \geq 0$.
Replace x_i by $-x_i^-$ when $x_i \leq 0$, and add $x_i^- \geq 0$.

$$
\begin{aligned}
\min\ & Z = 2x_1^+ - 2x_1^- - 3x_2 + Ma_3 \\
\text{s.t.}\ & x_1^+ - x_1^- + 3x_2 \leq 9 \quad s_1 \\
& -2x_1^+ + 2x_1^- - 5x_2 \leq 6 \quad s_2 \\
& x_2 + a_3 \geq 1 \quad e_3 \\
& x_1^+, x_1^-, \quad x_2 \geq 0, a_3 \geq 0
\end{aligned}
$$

Step 5 Convert LP to standard form using slack/excess variables.

$$
\begin{aligned}
\min\ & Z = 2x_1^+ - 2x_1^- - 3x_2 + Ma_3 \\
\text{s.t.}\ & x_1^+ - x_1^- + 3x_2 + s_1 = 9 \\
& -2x_1^+ + 2x_1^- - 5x_2 + s_2 = 6 \\
& x_2 - e_3 + a_3 = 1 \\
& x_1^+, x_1^-, \quad x_2 \geq 0, a_3 \geq 0, s_1, s_2, e_3 \geq 0
\end{aligned}
$$

Step 6 Use slack and artificial vars in the starting bfs, convert tableau to canonical form.

Proceed with subsequent steps of tableau simplex method.

$$\min\ z = 2x_1^+ - 2x_1^- - 3x_2 + Ma_3$$

s.t.

$$x_1^+ - x_1^- + 3x_2 + s_1 = 9$$

$$-2x_1^+ + 2x_1^- - 5x_2 + s_2 = 6$$

$$x_2 - e_3 + a_3 = 1$$

all vars ≥ 0

BV	z	x_1^+	x_1^-	x_2	s_1	s_2	e_3	a_3	rhs	
	1	−2	2	3	0	0	0	−M	0	$R_0 + MR_3$
	0	1	−1	3	1	0	0	0	9	
	0	−2	2	−5	0	1	0	0	6	
	0	0	0	1	0	0	−1	1	1	
	1	−2	2	M+3	0	0	−M	0	M	
s_1	0	1	−1	3	1	0	0	0	9	
s_2	0	−2	2	−5	0	1	0	0	6	
a_3	0	0	0	1	0	0	−1	1	1	
	1	−2	2	0	0	0	3	−(M+3)	−3	$R_0 - (M+3)R_3$
s_1	0	1	−1	0	1	0	3	−3	6	
s_2	0	−2	2	0	0	1	−5	5	11	
x_2	0	0	0	1	0	0	−1	1	1	
	1	−3	3	0	−1	0	0	−M	−9	
e_3	0	1/3	−1/3	0	1/3	0	1	−1	2	
s_2	0	−1/3	1/3	0	5/3	1	0	0	21	
x_2	0	1/3	−1/3	1	1/3	0	0	0	3	
	1	0	0	0	−16	−9	0	−M	−198	
e_3	0	0	0	0	2	1	1	−1	23	
x_1^-	0	−1	1	0	5	3	0	0	63	
x_2	0	0	0	1	2	1	0	0	24	

optimal solution: $x_1 = x_1^+ - x_1^- = 0 - 63 = -63$, $x_2 = 24$, $e_3 = 23$; $z^* = -198$.