MATH 364: Lecture 14 (10/03/2024)

Today: * Review for midterm

Optimal Solutions and # Optimal BFS's

Recall bfs ≡ corner point

We note the following points:

* an **optimal solution** is any feasible point that is optimal; it may or may not be a corner point, i.e., it may or may not be a bfs.
* But if the LP has a unique optimal solution, then that optimal solution must be a bfs.
* If the LP has alternative optimal solutions (case 2), it must have **infinitely many optimal solutions**.
* The total # bfs's possible is finite, since we can have at most $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ choices for the set of basic variables, but not all of them may give a bfs. Hence the **# optimal bfs's** is also **finite**.

Hw 6, Problem 5

You will get 3 optimal bfs's. Describe them as 4-vectors in the form
$$\bar{u} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \bar{v} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \quad \text{and} \quad \bar{w} = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}, \text{ say.}$$

Then describe all optimal solutions as convex combinations of $\bar{u}$, $\bar{v}$, and $\bar{w}$.

Hw 6, Problem 3

Alternative Optimal solutions exist if there is a non-basic variable with coefficient 0 in Row-0.

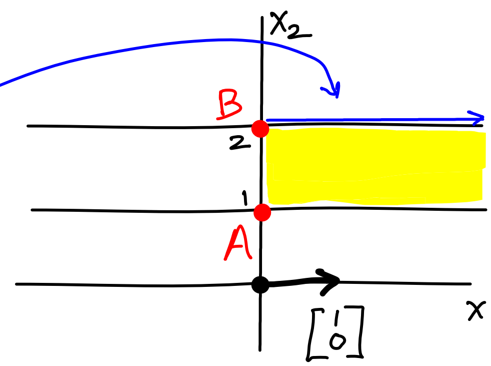
Here, x_1 is non-basic with 0 in its Row-0.

$\Rightarrow$ alternative optimal solutions exist.

BV	z	x_1	x_2	x_3	x_4	rhs
	1	0	0	0	4	5
x_3	0	−2	0	1	2	3
x_2	0	−3	1	0	1	1

But there are no candidates for min ratio test $\Rightarrow$ x_1 cannot enter the basis. So, there are no alternative optimal bfs's.

Consider $\left\{ \begin{array}{l} \max \ x_2 \\ \text{s.t.} \ 1 \le x_2 \le 2 \\ \quad x_1, x_2 \ge 0 \end{array} \right\}$. $z^* = 2$ here.



There is one optimal bfs at $B(0,2)$ and infinitely many optimal solutions.

All optimal solutions can be given as $\bar{x} = \begin{bmatrix}0\\2\end{bmatrix} + \alpha \begin{bmatrix}1\\0\end{bmatrix}$, $\alpha \ge 0$, i.e.,

$$\bar{x} = \begin{bmatrix}\alpha\\2\end{bmatrix}, \ \alpha \ge 0.$$

Hw 6, Problem 4

What happens if we were to pivot x_1 in?

We come to the conclusion that the LP is unbounded after that one pivot.

	z	x_1	x_2	x_3	x_4	rhs
	1	-5	0	0	-1	5
x_3	0	2	0	1	0	4
x_2	0	3	1	0	-2	6
	1	0	0	5/2	-1	15
x_1	0	1	0	1/2	0	2
x_2	0	0	1	$-3/2$	-2	0

LP is unbounded

But we can get to that conclusion without pivoting x_1 in!

Problems from Practice Midterm

1.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	c_1	c_2	0	$c_3=0$	c_4	$c_5=0$	z^*
0	3	a_1	1	0	a_2	$a_3=0$	1
0	-1	-2	0	$a_4=1$	-1	$a_5=0$	$b\geq 0$
0	a_6	-4	0	0	-3	$a_7=1$	3

x_1, x_2, x_5 cannot be basic (they must be unit vector columns).

So, x_3, x_4, x_6 must be basic.

x_3 is basic in Row-1.

$\boxed{c_3=0,\ a_4=1,\ c_5=0,\ a_3=0,\ a_5=0,\ a_7=1 \text{ and } b\geq 0 \text{ always hold.}}$

x_4 coefficient in Row 3 is $0 \Rightarrow x_4$ is basic in Row 2, and hence x_6 is basic in Row 3.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	$c_1 = 0$	$c_2 = 0$	0	$c_3 = 0$	$c_4 = 0$	$c_5 = 0$	z^*
0	3	$a_1 > 0$	1	0	$a_2 > 0$	$a_3 = 0$	1
0	-1	-2	0	$a_4 = 1$	-1	$a_5 = 0$	$b \geq 0$
0	a_6	-4	0	0	-3	$a_7 = 1$	3

(a) The current solution is optimal, and there are alternative optimal basic feasible solutions.

optimality: $c_1 \leq 0$, $c_2 \leq 0$, $c_4 \leq 0$

For alternative optimal bfs, we should be able to pivot a non-basic var with Row-0 coeff $= 0$ into the basis.

$(c_1 = 0)$ OR $(c_2 = 0, a_1 > 0)$ OR $(c_4 = 0, a_2 > 0)$, or any combinations of above settings.

(b) LP unbounded.

z	x_1	x_2	x_3	x_4	x_5	x_6	rhs
1	c_1	$c_2 > 0$	0	$c_3 = 0$	$c_4 > 0$	$c_5 = 0$	z^*
0	3	$a_1 \leq 0$	1	0	$a_2 \leq 0$	$a_3 = 0$	1
0	-1	-2	0	$a_4 = 1$	-1	$a_5 = 0$	$b \geq 0$
0	a_6	-4	0	0	-3	$a_7 = 1$	3

$(c_2 > 0$ and $a_1 \leq 0)$ OR $(c_4 > 0$ and $a_2 \leq 0)$, or both.

3. Careful about sign restrictions!
Plot at least one z-line.

4.

z	x_1	x_2	x_3	s_1	s_2	rhs
1	0	−3	0	−3	−1/2	−20
0	1	−1	0	1	−1	2
0	0	2	1	0	1/2	2
1	3	−6	0	0	−7/2	−14
0	1	−1	0	1	−1	2
0	0	2	1	0	1/2	2
1	3	8	7	0	0	0
0	1	3	2	1	0	6
0	0	4	2	0	1	4

min LP, as all #s in Row-0 (under vars) are ≤ 0.

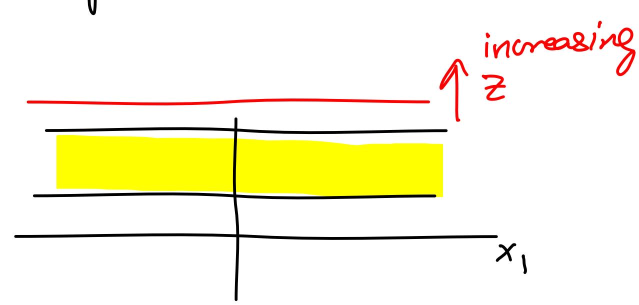
$$\min z = -3x_1 - 8x_2 - 7x_3$$
s.t.
$$x_1 + 3x_2 + 2x_3 \leq 6$$
$$4x_2 + 2x_3 \leq 4$$
$$x_1,\ x_2,\ x_3 \geq 0$$

5. T/F

a) feasible region has no corner points.

FALSE

$\max\ x_2$
s.t. $1 \leq x_2 \leq 2$
x_1 urs



has no corner points, but $z^* = 2$ (i.e., not unbounded LP)

b) False.
If min ratio $= 0$, z does not change.