MATH 364: Lecture 17 (10/15/2024)

* Submit anonymous feedback @ any time!
* You're welcome to use Matlab/Python to show work for tableau simplex in homework.

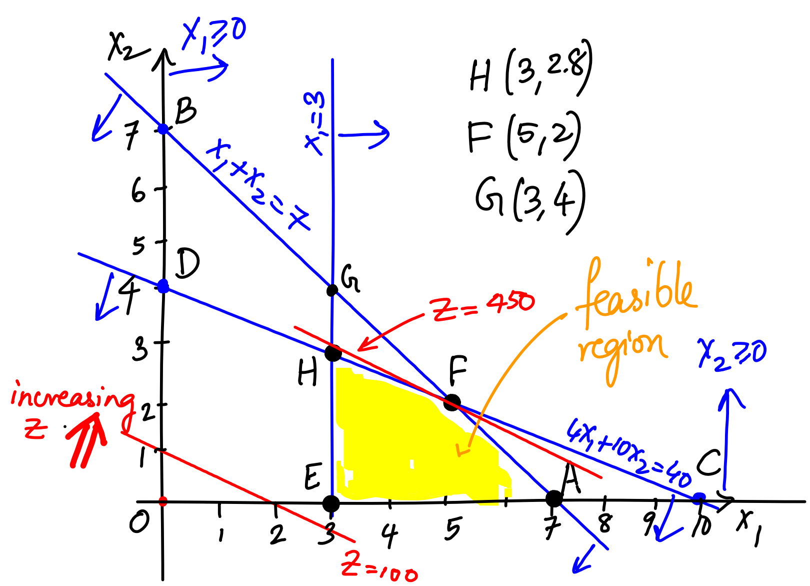
Today: * sensitivity analysis
- change in c_2 (coefficient of x_2)
- change in b_i, shadow price

Changing revenue/acre of wheat

First, let's assume price/bushel of corn is \$5. So the objective function is max $z = 50x_1 + 100x_2$. Now, F(5,2) is optimal, with $z^* = 450$. The analysis becomes more interesting here, as compared to the original Farmer Jones LP.

$$
\begin{aligned}
\max \quad & z = \cancel{30}^{50}x_1 + 100x_2 && \text{(total revenue)} \\
\text{s.t.} \quad & x_1 + x_2 \le 7 && \text{(land availability)} \\
& 4x_1 + 10x_2 \le 40 && \text{(labor hrs)} \\
& 10x_1 \ge 30 && \text{(min corn)} \\
& x_1, x_2 \ge 0 && \text{(non-negativity)}
\end{aligned}
$$

Optimal solution is at F(5,2), with $z^* = 450$.



Q. For what values of revenue/acre of wheat (c_2; $=100$ now) is the current solution F(5,2) optimal?

Let c_2 be the coefficient of x_2 in the objective function.

Slope of z-line is $\frac{-50}{c_2}$. $\qquad (z = 50x_1 + c_2 x_2)$

Both the (labor-hrs) and the (land-available) constraints are binding at the current optimal solution (at $F(5,2)$). This solution remains optimal as long as the slope of the z-line is in between the slopes of the two binding constraints at F.

Thus, current solution remains optimal as long as

$$\frac{-1}{1} \le \frac{-50}{c_2} \le \frac{-4}{10}$$

We invert the three fractions and scale the two inequalities by -1, and hence the inequality senses stay as $\le$.

$$\Rightarrow \quad 1 \le \frac{c_2}{50} \le \frac{10}{4} \quad \Rightarrow \quad 50 \le c_2 \le 125.$$

Equivalently, if the price per bushel of wheat is \$w, then for $\frac{50}{25} \le w \le \frac{125}{25}$, i.e., $2 \le w \le 5$, the current solution remains optimal.

\# bushels/acre of wheat.

e.g., if $w = \$3.5$, $x_1 = 5$, $x_2 = 2$ is still optimal, but new
$z^* = 50x_1 + 3.5 \times 25 \times 2 = \$425.$

Change in RHS Coefficient (Stay with $z = 50x_1 + 100x_2$).

max $z = \cancel{30}^{50} x_1 + 100x_2$ (total revenue)

s.t.

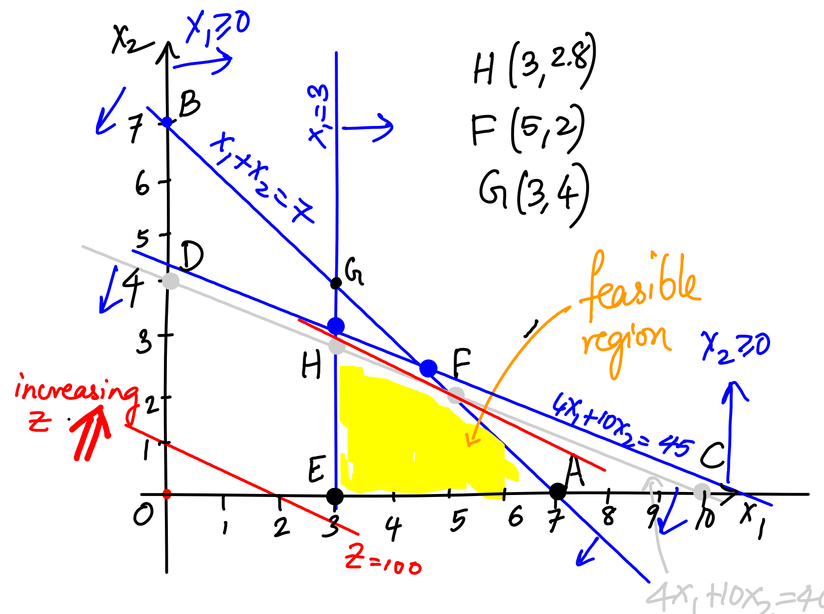
$x_1 + x_2 \le 7$ (land availability)

$4x_1 + 10x_2 \le 40$ (labor hrs)

$10x_1 \ge 30$ (min corn)

$x_1, x_2 \ge 0$ (non-negativity)

Optimal solution is at $F(5,2)$, $z^* = 450$.



Q. What happens if Jones has 45 hrs of labor/wk?

It appears the optimal basis remains same ($x_1 > 0$ and $x_2 > 0$) as long as the (labor hrs) line moves parallel to itself between A(7,0) and G(3,4).

By sliding the (labor hrs) line up/down, we see that between 28 and 52 hrs/week, the current basis remains optimal.

What is the effect of changing b_i on z^* and $\bar{x}^*$?

optimal solution

optimal objective fn. value

Let us assume b_2 (here) changes from 40 to $40 + \Delta$.

As F remains optimal for $28 \le b_2 \le 52$, we get

$$28 \le 40 + \Delta \le 52 \implies -12 \le \Delta \le 12.$$

range of values of Δ for which current basis remains optimal.

How did we get the limits of 28 and 52? Here are some details.

First, notice that as long as the (land) and (labor.hrs) constraints remain binding, the current basis, given by $BV = \{x_1, x_2, e_3\}$ will remain optimal, even if z^* and the values of x_1, x_2 might be different.

Notice that (min.corn) constraint is non-binding at $F(5,2)$, and hence $e_3 > 0$, making it basic.
$e_3 \longrightarrow$ excess var for (min corn) constraint

In general, let the rhs value of the (labor hrs) constraint be b_2. We can ask: "For what values of b_2 is the current basis ($BV = \{x_1, x_2, e_3\}$) optimal?"

As long as $4x_1 + 10x_2 = b_2$ is (at or) below G and (at or) above A, current basis remains optimal.

At $A(7,0)$, $b_2 = 4 \times 7 + 10 \times 0 = 28$, and
at $G(3,4)$, $b_2 = 4 \times 3 + 10 \times 4 = 52$.

Hence, for $28 \le b_2 \le 52$, the current basis is optimal.

New $z^*, \bar{x}^*$? New F, i.e., F_Δ, (and $\bar{x}^*_\Delta$)

$$x_1 + x_2 = 7 \quad \text{---} (1)$$
$$4x_1 + 10x_2 = 40 + \Delta \quad \text{---} (2)$$

(2)−4(1): $6x_2 = 12 + \Delta \Rightarrow x_2 = 2 + \frac{\Delta}{6}$

so, (1) $\Rightarrow x_1 = 5 - \frac{\Delta}{6}$. So $F_\Delta = \left(5 - \frac{\Delta}{6}, 2 + \frac{\Delta}{6}\right). (= \bar{x}^*_\Delta)$

$$\Rightarrow z^*_\Delta = 50x_1 + 100x_2 = 50\left(5 - \frac{\Delta}{6}\right) + 100\left(2 + \frac{\Delta}{6}\right)$$
$$= 450 + \left(\frac{50}{6}\right)\Delta = 450 + \left(\frac{25}{3}\right)\Delta.$$

shadow price of (labor-hrs).

Def The **shadow price** of constraint i is the amount by which the value of z improves for a unit increase in b_i (rhs) provided the optimal basis remains same after the increase.

Economic interpretation

Shadow price here is the price that Jones is willing to pay to get an extra hour of labor. For instance, if he can get someone to work 5 hrs extra for \$6/hr, he will take it, as his total revenue will increase by $\$\frac{25}{3} = \$8.33$ for each extra labor hour, resulting in a net profit.

Shadow price of land constraint?

Go back to the original problem with $4x_1 + 10x_2 \le 40$, but now change b_1 in $x_1 + x_2 \le b_1$ so that F still remains optimal. We get $5.8 \le b_1 \le 10$ as the range.

x_1+x_2 @ $H(3, 2.8)$ $\qquad$ x_1+x_2 @ $C(10,0)$

Let b_1 change from 7 to $7+\Delta$. The new optimal solution F is

$$x_1 + x_2 = 7+\Delta \quad \text{——— (1)}$$
$$4x_1 + 10x_2 = 40 \quad \text{——— (2)}$$

$$(2) - 4(1) \Longrightarrow \quad 6x_2 = 12 - 4\Delta \quad \Rightarrow \quad x_2 = 2 - \tfrac{2}{3}\Delta.$$

So, (1) $\Rightarrow \quad x_1 = 7+\Delta - (2 - \tfrac{2}{3}\Delta) = 5 + \tfrac{5}{3}\Delta.$

i.e., $F\left(5+\tfrac{5}{3}\Delta,\ 2-\tfrac{2}{3}\Delta\right)$ is the new optimal solution, and

$$z^* = 50x_1 + 100x_2 = 50\left(5+\tfrac{5}{3}\Delta\right) + 100\left(2 - \tfrac{2}{3}\Delta\right)$$
$$= 450 + \tfrac{50}{3}\Delta.$$

$\Rightarrow$ Shadow price of acres constraint is $\frac{50}{3}$. In other words, Jones would be willing to pay up to $\$\frac{50}{3}$ for one extra acre of land.

Shadow price of the (min corn) constraint is zero, as $30 \to 30+\Delta$ will not change the optimal solution $F(5,2)$.

Shadow price of a non-binding constraint is always zero!