MATH 364: Lecture 18 (10/17/2024)

Today:
* Tableau simplex in Matlab
* simplex method in matrix form

You can do the computations (EROs) for the tableau simplex method in Matlab (or Python). See the web page for an example.

Recall Sensitivity analysis in 2D. We have seen:

* Change in objective function coefficients (c_j).
  - Q.1. For what range of values of c_j does the current optimal solution remain optimal?
  - 2. What if c_j changes beyond this range? Could we reoptimize quickly? → we'll address such questions about reoptimization now...

* Change in the right-hand side (rhs) coefficients (b_i).
  - Q1. For what range of values of b_i does the current basis remain optimal?
  - 2. What if b_i changes beyond this range? Could we reoptimize quickly?

Now, we want to generalize to n-dimensions. We also want to consider
* change in constraint coefficients (A_{ij})
* Adding a column ...

In order to consider sensitivity analysis in higher dimensions, we first consider the simplex method in matrix form. This way, we will formalize many more properties of the tableau simplex method.

Simplex Method in Matrix Form

$$\left.\begin{array}{ll} \max & z=\bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x}=\bar{b} \\ & \bar{x}\geq\bar{0} \end{array}\right\}$$ LP in standard form, after adding slack/excess/artificial vars.

A is $m\times n$, $\bar{b}$ is an m-vector, $\bar{c}$ and $\bar{x}$ are n-vectors.
$\bar{c}$ contains $-M$ terms for artificial variables.

$$\max \quad z = \overbrace{\begin{bmatrix} \text{---} \ \bar{c}^T \ \text{---} \end{bmatrix}}^{n} \left.\begin{bmatrix} | \\ \bar{x} \\ | \end{bmatrix}\right\} n$$

$\bar{c}^T$: transpose of $\bar{c}$

$$\text{s.t.} \quad m\updownarrow \overbrace{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}}^{n} \begin{bmatrix} | \\ \bar{x} \\ | \end{bmatrix} = \begin{bmatrix} | \\ \bar{b} \\ | \end{bmatrix}$$

$$\bar{x}\geq\bar{0}$$

Suppose we know which variables are basic in the optimal solution.
Let $\bar{x}_B$ denote all the basic variables, and $\bar{x}_N$ the non-basic variables.

$$n\updownarrow \begin{bmatrix} \\ \bar{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \bar{x}_B \\ \hline \bar{x}_N \end{bmatrix} \begin{array}{l} \updownarrow m \\ \updownarrow n-m \end{array}$$

after possible reordering of the variables. Of course, there is no harm in reordering the x_j's, since order of addition is immaterial.

We split A and $\bar{c}$ in the same fashion into basic and non-basic parts.

$$\max \quad z = \overbrace{\left[\, \bar{c}_B^T \mid \bar{c}_N^T \,\right]}^{n} \left.\begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix}\right\} n \qquad\qquad z = \bar{c}_B^T \bar{x}_B + \bar{c}_N^T \bar{x}_N$$

s.t.

$$m \updownarrow \overbrace{\left[\; B \;\middle|\; N \;\right]}^{n} \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix} = \begin{bmatrix} \bar{b} \end{bmatrix} \qquad\qquad B\bar{x}_B + N\bar{x}_N = \bar{b}$$

$$\bar{x} \geq \bar{0}$$

Starting tableau T_0

	z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	1	$-\bar{c}_B^T$	$-\bar{c}_N^T$	0
m	$\bar{0}$	B	N	$\bar{b}$

$\longleftarrow n+2 \longrightarrow$

$\xrightarrow[\text{or EROs}]{\text{simplex method}}$

T^*: optimal tableau

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	0	?	?
$\bar{0}$	I_m	?	?

T_0: $(m+1) \times (n+2)$

Recall: How to invert an $m \times m$ matrix A:

$$[A \mid I_m] \xrightarrow{\text{EROs}} [I_m \mid A^{-1}]$$

We want to find the inverse of the $(m+1) \times (m+1)$ "basic" matrix

$$\left(\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}^{-1} \right) \underbrace{\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}}_{\text{"basic" matrix}} = \begin{bmatrix} 1 & 0 \\ \bar{0} & I_m \end{bmatrix}$$

The basis matrix B itself is invertible by definition, i.e., B^{-1} exists.

We claim
$$\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1}\end{bmatrix}.$$

Let's check!
$$\begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1}\end{bmatrix}\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B\end{bmatrix} = \begin{bmatrix} 1\times 1 + \bar{c}_B^T B\cdot\bar{0} & 1\times -\bar{c}_B^T + \bar{c}_B^T \overbrace{B^{-1}B}^{I_m} \\ \bar{0}\times 1 + B^{-1}\bar{0} & \bar{0}\cdot -\bar{c}_B^T + \underbrace{B^{-1}B}_{I_m}\end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \bar{0} & I_m\end{bmatrix}.$$

you should check the other multiplication, i.e., $\begin{bmatrix} 1 & -\bar{c}_B^T \\ \bar{0} & B\end{bmatrix}\begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1}\end{bmatrix} = I_{m+1}$, as well.

So, T^* is given by

$$\begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1}\end{bmatrix} T_0 = \begin{bmatrix} 1 & \bar{c}_B^T B^{-1} \\ \bar{0} & B^{-1}\end{bmatrix} \begin{array}{c|c|c|c} z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline 1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\ \hline \bar{0} & B & N & \bar{b} \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|c} z & \bar{x}_B & \bar{x}_N & \text{rhs} \\ \hline 1 & 0 & -\bar{c}_N^T + \bar{c}_B^T B^{-1} N & \bar{c}_B^T B^{-1}\bar{b} \\ \hline \bar{0} & I_m & B^{-1}N & B^{-1}\bar{b} \\ \hline \end{array} = T^* \quad \text{(optimal tableau)}$$

Hence, $z^* = \bar{c}_B^T B^{-1}\bar{b}$, and the optimal solution is given by

$$\bar{x}_B^* = B^{-1}\bar{b}, \quad \text{i.e.,} \quad \bar{x}^* = \begin{bmatrix} \bar{x}_B^* \\ \bar{x}_N^* \end{bmatrix} = \begin{array}{c} \begin{bmatrix} B^{-1}\bar{b} \\ \bar{0} \end{bmatrix} \end{array} \begin{array}{l} \updownarrow m \\ \updownarrow n-m \end{array}.$$

Hence, if we know which variables are going to be basic in the optimal tableau, we could convert the starting tableau to the optimal tableau through direct matrix multiplication.

Also, we can read off the elementary matrix from the optimal tableau!

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	$\bar{0}$	$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N$	$\bar{c}_B^T B^{-1} \bar{b}$
$\bar{0}$	I	$B^{-1} N$	$B^{-1} \bar{b}$

If $N = I$, then the columns under $\bar{x}_N$ are

$$-\bar{c}_N^T + \bar{c}_B^T B^{-1}$$

$$B^{-1}$$

Further, if $\bar{c}_N = \bar{0}$, we have the elementary matrix!

In particular, we can read off B^{-1} from under the columns of slack and artificial variables in rows 1 to m in the optimal tableau!

We now look at an LP for which we write down the optimal tableau "directly" when the optimal basis is given. Our goal in studying the simplex method in this matrix form is to be able to do sensitivity analysis in a general form.