MATH 364: Lecture 19 (10/22/2024)

Today: * sensitivity analysis in matrix form
- changing c_j when x_j is nonbasic
- changing c_j when x_j is basic

Recall: Simplex Method in matrix form:

starting tableau

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	$-\bar{c}_B^T$	$-\bar{c}_N^T$	0
$\bar{0}$	B	N	$\bar{b}$

$\xrightarrow{\text{EROs}}$

optimal tableau

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	$\bar{0}$	$-\bar{c}_N^T+\bar{c}_B^TB^{-1}N$	$\bar{c}_B^TB^{-1}\bar{b}$
$\bar{0}$	I_m	$B^{-1}N$	$B^{-1}\bar{b}$

$$\begin{aligned}
\max\ z = -x_1+x_2 & \\
\text{s.t.}\quad 2x_1+x_2 &\leq 4 \quad s_1\\
x_1+x_2 &\leq 2 \quad s_2\\
x_1, x_2 &\geq 0
\end{aligned}$$

We are given $\{x_2, s_1\}$ in that order are optimal. Find the optimal tableau.

$BV=\{z, x_2, s_1\}$, $NBV=\{x_1, s_2\}$.

Starting tableau

z	x_2	s_1	x_1	s_2	rhs
1	-1	0	1	0	0
0	1	1	2	0	4
0	1	0	1	1	2

($-\bar{c}_B^T$ and B under x_2, s_1; $-\bar{c}_N^T$ and N under x_1, s_2; $\bar{b}$ in rhs)

optimal tableau

z	x_2	s_1	x_1	s_2	rhs
1	0	0	2	1	2
0	1	0	1	1	2
0	0	1	1	-1	2

($-\bar{c}_N^T+\bar{c}_B^TB^{-1}N$ and $B^{-1}N$ under x_1, s_2; $\bar{c}_B^TB^{-1}\bar{b}$ and $B^{-1}\bar{b}$ in rhs)

$$\bar{c}^T = \begin{array}{cccc} x_2 & s_1 & x_1 & s_2 \\ \end{array} \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \end{array}\right]$$

$$-\bar{c}_B^T = \begin{array}{cc} x_2 & s_1 \end{array} \left[\begin{array}{cc} -1 & 0 \end{array}\right]$$

$$-\bar{c}_N^T = \begin{array}{cc} x_1 & s_2 \end{array} \left[\begin{array}{cc} 1 & 0 \end{array}\right]$$

$$B = \begin{array}{c} \begin{array}{cc} x_2 & s_1 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array} \Rightarrow B^{-1} = \frac{1}{-1}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow B^{-1}N = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\bar{c}_B^TB^{-1}N = [1\ \ 0]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [1\ \ 1],$$

$$-\bar{c}_N^T+\bar{c}_B^TB^{-1}N = [1\ \ 0]+[1\ \ 1] = [2\ \ 1],$$

$$B^{-1}\bar{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and}$$

$$\bar{c}^TB^{-1}\bar{b} = [1\ \ 0]\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2$$

We now consider sensitivity analysis using the matrix form of the simplex method. In preparation, we first write down the entries in the column of a variable x_j in the optimal tableau.

$$
\begin{array}{cccc}
z & \bar{x}_B & \bar{x}_N & \text{rhs} \\
\hline
1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\
\hline
\bar{0} & B & N & \bar{b} \\
\hline
\end{array}
\xrightarrow{\text{EROs}}
\begin{array}{cccc}
\multicolumn{4}{c}{\text{optimal tableau}} \\
z & \bar{x}_B & \bar{x}_N & \text{rhs} \\
\hline
1 & 0 & -\bar{c}_N^T + \bar{c}_B^T B^{-1} N & \bar{c}_B^T B^{-1} \bar{b} \\
\hline
0 & I_m & B^{-1}N & B^{-1}\bar{b} \\
\hline
\end{array}
$$

Column of x_j in the optimal tableau:

where $\bar{a}_j$ is the column of x_j in A.

$$
\begin{array}{c}
x_j \\
\hline
-c_j + \bar{c}_B^T B^{-1} \bar{a}_j \\
\hline
B^{-1}\bar{a}_j \\
\hline
\end{array}
$$

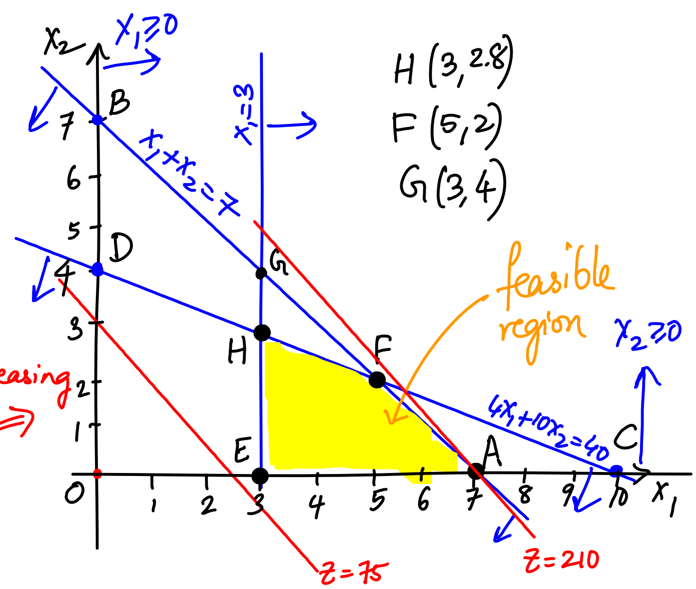
This form applies for both non-basic and basic x_j's. If x_j is basic in Row-i, then $B^{-1}\bar{a}_j$ will be $\bar{e}_i$, the i^{th} m-unit vector. Also, $-c_j + \bar{c}_B^T B^{-1} \bar{a}_j = -c_j + c_j = 0$.

1. Changing c_j when x_j is non-basic

We change revenue/acre of wheat to \$25, so that x_2 is nonbasic at the optimal solution (and x_1 is indeed basic, which we will use in the next type of sensitivity analysis).

$$
\begin{array}{lrcll}
\max & z = 30x_1 + 25x_2 & & & \text{(revenue)} \\
\text{s.t.} & x_1 + x_2 & \le & 7 & \text{(land avail.) } s_1 \\
 & 4x_1 + 10x_2 & \le & 40 & \text{(labor hrs) } s_2 \\
 & \cancel{10}x_1 & \ge & 3\cancel{0} & \text{(min corn) } e_1 \\
 & x_1, x_2 & \ge & 0 & \text{(nonneg)}
\end{array}
$$

Can scale by 10 to get $x_1 \ge 3$.



The optimal solution is at $A(7, 0)$, with $z^* = 210$.

Tableau simplex:

BV	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	−30	−25	0	0	0	M	0
s_1	0	1	1	1	0	0	0	7
s_2	0	4	10	0	1	0	0	40
a_3	0	1	0	0	0	−1	1	3
	1	−M−30	−25	0	0	M	0	−3M
s_1	0	1	1	1	0	0	0	7
s_2	0	4	10	0	1	0	0	40
a_3	0	1	0	0	0	−1	1	3
	1	0	−25	0	0	−30	M+30	90
s_1	0	0	1	1	0	1	−1	4
s_2	0	0	10	0	1	4	−4	28
x_1	0	1	0	0	0	−1	1	3
	1	0	5	30	0	0	M	210
e_3	0	0	1	1	0	1	−1	4
s_2	0	0	6	−4	1	0	0	12
x_1	0	1	1	1	0	0	0	7

identity matrix under columns of s_1, s_2, a_3

$R_0 - M R_3$

$R_0 + (M+30) R_3$

B^{-1} is sitting in the columns that had I_3 in the starting tableau.

Optimal basis is $\{e_3, s_2, x_1\}$ in that order ($\equiv A(7,0)$).
So, x_2 is non-basic.

We can now write down the components of the optimal tableau as just described, i.e., $\bar{C}_B, \bar{C}_N, B^{-1}, B^{-1}\bar{b}, B^{-1}N$, etc.

how did we find B^{-1}?!

$$\bar{C}_B^T = \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 30 \end{bmatrix}_{1\times 3} \end{array} \qquad B^{-1} = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3} \end{array}$$

$$\Rightarrow \quad \bar{C}_B^T B^{-1} = [30 \quad 0 \quad 0].$$

We have I_3 (3x3 identity matrix) under the columns of s_1, s_2, a_3 in the starting tableau. And hence B^{-1} is sitting under these columns in the optimal tableau.

Recall: We have $B^{-1}N$ in the optimal tableau. Thus, if a submatrix of N is I (identity matrix), that submatrix will have B^{-1} in the optimal tableau. More generally, if a submatrix of A is I, then that submatrix is converted to B^{-1} in the optimal tableau.

Let's check to make sure B^{-1} is indeed correct. First, notice

$$B = \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \end{array},$$ the columns of e_3, s_2, x_1 from A, in that order.

Hence $$B^{-1}B = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$ and

similarly, $$BB^{-1} = \begin{array}{c} \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

You're welcome to use a package such as Octave (Matlab) or Python to do these matrix calculations. But you will not be tested on the use of such software package(s).

Suppose coefficient of x_2 in the objective function changes to $25+\Delta$.

Questions 1. For what range of values of Δ does the current basis remain optimal?

2. If for some Δ, the current basis is not optimal, how do we find the new optimal basis and solution (quickly)?

without starting from scratch, and resolving the LP all over again.

With $c_2 = 25+\Delta$, the entries in the x_2-column are

$$\begin{array}{c} x_2 \\ \hline -c_2+\bar{c}_B^T B^{-1}\bar{a}_2 \\ \hline B^{-1}\bar{a}_2 \\ \hline \end{array} \rightarrow \begin{array}{c} x_2 \\ \hline -(25+\Delta) + [30\ 0\ 0]\begin{bmatrix}1\\10\\0\end{bmatrix} \\ \hline \begin{bmatrix}1 & 0 & -1\\ -4 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\10\\0\end{bmatrix} \\ \hline \end{array} \rightarrow \begin{array}{c} x_2 \\ \hline 5-\Delta \\ \hline 1 \\ 6 \\ 1 \\ \hline \end{array}$$

Current basis remains optimal as long as $5-\Delta \geq 0$, i.e., $\Delta \leq 5$

"reduced cost" of wheat

The current solution remains optimal as well for $\Delta \leq 5$.

Def The **reduced cost** of a non-basic variable (in a max-LP) is the maximum amount by which its objective function coefficient can be increased with the current basis remaining optimal.

If the objective function coefficient of a nonbasic variable increases by more than its reduced cost the variable can enter the basis, and improve the value of z. At this point, the current basis becomes suboptimal. Here, we could pivot this non-basic variable into the basis from the current optimal tableau (and not start from scratch again).

Consider $\Delta = 7$ here, for instance. We could pivot x_2 into the basis, and obtain the new optimal tableau in one (new) pivot.

	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	0	$\underbrace{-2}_{5-\Delta}$	30	0	0	M	210
e_3	0	0	1	1	0	1	−1	4
s_2	0	0	6	−4	1	0	0	12
x_1	0	1	1	1	0	0	0	7
	1	0	0	86/3	1/3	0	M	214
e_3	0	0	0	5/3	−1/6	1	−1	2
x_2	0	0	1	−2/3	1/6	0	0	2
x_1	0	1	0	5/3	−1/6	0	0	5

New $z^* = 214$ (at $F(5,2)$).

Notice that once the revenue/acre of wheat is \$32, which is higher than the revenue/acre of corn (still at \$30), it makes sense to farm both wheat and corn.

2. Changing c_j when x_j is basic

Consider changing c_1 (coefficient of x_1) from 30 to $30+\Delta$. Since an entry in $\bar{C}_B^T$ is changing here, more entries in Row-0 under the non-basic columns could change as compared to the case when we were changing a non-basic c_j.

Now,
$$\bar{C}_B^T = \begin{array}{ccc} e_3 & s_2 & x_1 \\ \end{array}\left[\begin{array}{ccc} 0 & 0 & 30+\Delta \end{array}\right]_{1\times 3} \qquad B^{-1} = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{3\times 3} \end{array},$$
and hence

$$\bar{C}_B^T B^{-1} = [30+\Delta \quad 0 \quad 0].$$

Current basis remains optimal as long as $c_j' \geq 0$ for all j (i.e., the numbers in Row-0 remain ≥ 0).

$c_j' = 0$ if x_j is basic, and hence we concentrate on the non-basic entries.

For the non-basic variables,

$$-\bar{C}_N^T + \bar{C}_B^T B^{-1} N = \begin{array}{ccc} x_2 & s_1 & a_3 \end{array} \atop [-25 \quad 0 \quad M] + [30+\Delta \quad 0 \quad 0] \begin{array}{c} \begin{array}{ccc} x_2 & s_1 & a_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= [-25 \quad 0 \quad M] + [30+\Delta \quad 30+\Delta \quad 0]$$

$$= [5+\Delta \quad 30+\Delta \quad M]$$

$\Rightarrow$ Current basis remains optimal as long as

$$[5+\Delta \quad 30+\Delta \quad M] \geq \bar{0}^T$$

$$\Rightarrow \left.\begin{array}{l} 5+\Delta \geq 0 \Rightarrow \Delta \geq -5 \\ 30+\Delta \geq 0 \Rightarrow \Delta \geq -30 \end{array}\right\} \Rightarrow \boxed{\Delta \geq -5.}$$

As long as revenue per acre of corn is at least \$25, which is the same as that for wheat, we continue to farm corn in all 7 acres.

If $\Delta = -8$, for instance, we can find the updated tableau (non-optimal) for that value of Δ, and continue the simplex method from there.