# MATH 364: Lecture 1 (08/20/2024)

This is Principles of Optimization.

I'm Bala Krishnamoorthy. Call me Bala.

In WSU since 2004, and have been in USA since 1999. I'm originally from India. If you do not understand what I say because of my accent, do let me know ☺!

My research interests are in optimization, algebraic topology, applications to biology, medicine, etc.

---

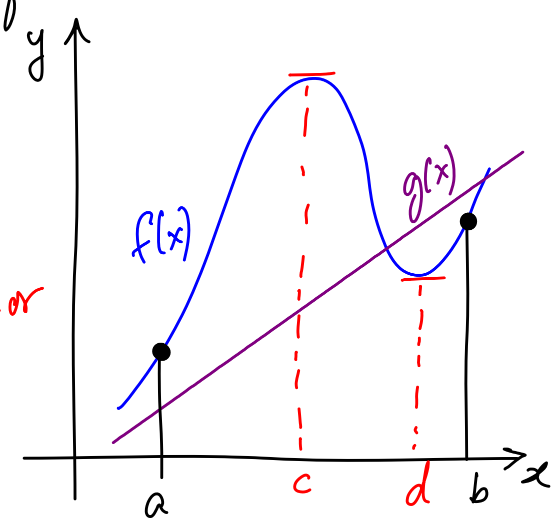
## Optimization — what is it?

In Calculus, you would've seen problems of the form $\min/\max f(x)$ for $a \le x \le b$.

$f'(x)=0$ gives critical points. In addition,

$f''(x) > 0$ gives minima
$f''(x) < 0$ gives maxima

these are local maxima or minima!




But we have to examine the **end points** of the interval as well!

Here, the minimum in the interval $[a,b]$ is at $x=a$, an end point.

**If $g(x)$ is linear, the maxima/minima are at the end points!**

In Math 364, we extend this easier linear case to higher dimensions.
If $g(x_1, x_2, \ldots, x_n)$ is linear, the optima still occur at corner points ($\equiv$ end points).

## A Motivating Problem

Dude M. Major ("Math") has a Thursday Problem.

Has 5 hrs, \$48 to spare

| | Costs | Utility |
|---|---|---|
| — can get tutoring | \$8/hr | 2/hr |
| — can party | \$16/hr | 3/hr |

How many hours to get tutored, and how many to party so that total utility is maximized?

Two decisions: $\begin{cases} x_1 = \text{\# hrs to get tutored} \\ x_2 = \text{\# hrs to party} \end{cases}$

objective/goal: maximize total utility while not exceeding the total hours and cash available.

$$\begin{array}{lrll} \max & 2x_1 + 3x_2 & & \text{(total utility)} \\ \text{s.t.} & x_1 + x_2 & \leq 5 & \text{(time available)} \\ & 8x_1 + 16x_2 & \leq 48 & \text{(cash available)} \\ & x_1, x_2 & \geq 0 & \text{(non-negativity)} \end{array}$$

"maximize"

"subject to"

linear optimization model/problem or linear program (LP).

We will go through LP formulation problems of this kind in detail.

If Dude could spend all time and money, we can write

$x_1 + x_2 = 5$
$8x_1 + 16x_2 = 48$

Ignore utility for now — it may well be not ideal to spend all money and time from a utility point-of-view. But we will come back to it later.

This is a system of linear equations of the form $A\bar{x} = \bar{b}$

vectors $(\bar{a}, \bar{b}, \bar{x}, \bar{\theta}, \text{etc.})$ → Bala's notation!

How do you solve $A\bar{x} = \bar{b}$ ? → Should've learned all about it in Math 220!

* form $[A|\bar{b}]$, the augmented matrix;
* use elementary row ops (EROs) to reduce $[A|\bar{b}]$ to echelon form and then to reduced row echelon form (RREF).

notation:
$R_i$ : $i^{th}$ row

EROs:

1. Exchange/swap row $\quad R_i \rightleftarrows R_j$
2. Scaling: multiply $R_i$ by $\alpha \neq 0 \quad \alpha R_i$
3. Replacement: $\quad R_i \leftarrow R_i + \alpha R_j \quad$ (or just $R_i + \alpha R_j$)
   replace row $i$ by the sum of itself and $\alpha$ times row $j$ ($\alpha \neq 0$ for nontrivial ERO).

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 16 & 48 \end{array}\right] \xrightarrow{R_2 - 8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 8 & 8 \end{array}\right] \xrightarrow{R_2\left(\frac{1}{8}\right)} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 - R_2}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$x_1 = 4,\ x_2 = 1$ is the unique solution

→ Dude has to study for 4 hrs and party for only 1 hr ☹!

This process of taking $[A|\bar{b}]$ to echelon form, and then to RREF is called Gaussian elimination, or the Gauss-Jordan method.

## Basic and Non-basic variables

Assume $A$ is $m \times n$ $(m \leq n)$ now (i.e., more general, not square).

$$[A|\bar{b}] \xrightarrow{\text{EROs}} [\tilde{A}|\tilde{b}] \longrightarrow \text{RREF}$$

After applying Gaussian elimination on $[A|\bar{b}]$ to get $[\tilde{A}|\tilde{b}]$, the variables that have a coefficient of 1 in one row and zero everywhere else are called basic variables (BV). All variables that are not basic are non-basic variables (NBV).

The system $A\bar{x}=\bar{b}$

1. has no solution if $[\tilde{A}|\tilde{b}]$ has a row of the form $[0\ 0 \cdots 0 | \tilde{b}_i \neq 0]$ (system is inconsistent)

2. If $[\tilde{A}|\tilde{b}]$ has no such inconsistent row then

   (a) if all variables are basic, then the system has a unique solution;

   (b) if there are free variables, the system has infinitely many solutions.

We had seen an instance of 2(a), giving $x_1=4$, $x_2=1$ as the unique solution.

Now assume that partying is also \$8/hr. Then we get

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 8 & 48 \end{array}\right] \xrightarrow{R_2 - 8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & 8 \end{array}\right] \rightarrow \text{inconsistent!}$$

This is an example of Case 1.

Moving on, assume the cash is \$40 now. We get

non-basic or free variable

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 8 & 40 \end{array}\right] \xrightarrow{R_2 - 8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$$

This system is an example of 2(b): infinitely many solutions.

Since we have many solutions, we could try to find one that maximizes total utility ($2x_1+3x_2$). Then Dude will party for all 5 hrs!

Once Dude insisted that he has to use up all the time and money, the objective (of maximizing total utility) did not play any role in finding the solution. The system has a unique solution in this case ($x_1=4$, $x_2=1$).

But in the last case, where there are infinitely many solutions, the objective will play a part. In this case, as long as $x_1+x_2=5$, and $x_1, x_2 \geq 0$, the solution is valid. Among all such solutions, we could pick the one that gives the largest value for $2x_1+3x_2$ (total utility). Hence $x_2=5$ (and $x_1=0$) gives the optimal solution here, i.e., Dude should just party all five hours!

We will learn (later) that these non-basic variables, which are also called free variables, are critical for solving linear optimization problems.