MATH 364 : Lecture 20 (10/24/2024)

Today: * change of b_i
* shadow price
* changing column of x_j

Recall: simplex method in matrix form:

$$
\begin{array}{c|cc|c}
z & \bar{x}_B & \bar{x}_N & \text{rhs} \\
\hline
1 & -\bar{c}_B^T & -\bar{c}_N^T & 0 \\
\hline
\bar{0} & B & N & \bar{b}
\end{array}
\xrightarrow{\text{EROs}}
\begin{array}{c|cc|c}
z & \bar{x}_B & \bar{x}_N \ \geq 0 & \text{rhs} \\
\hline
1 & 0 & -\bar{c}_N^T + \bar{c}_B^T B^{-1} N & \bar{c}_B^T B^{-1} \bar{b} \\
\hline
0 & I_m & B^{-1} N & B^{-1}\bar{b} \ \geq 0
\end{array}
$$

The current basis remains optimal as long as

1. $-\bar{c}_N^T + \bar{c}_B^T B^{-1} N \geq \bar{0}^T$ and (optimality for max-LP)
2. $B^{-1}\bar{b} \geq \bar{0}$. (feasibility)

3. Changing the right-hand side (rhs) of a constraint (b_i)

$b_i \longrightarrow b_i + \Delta$, so only rhs column is changed.

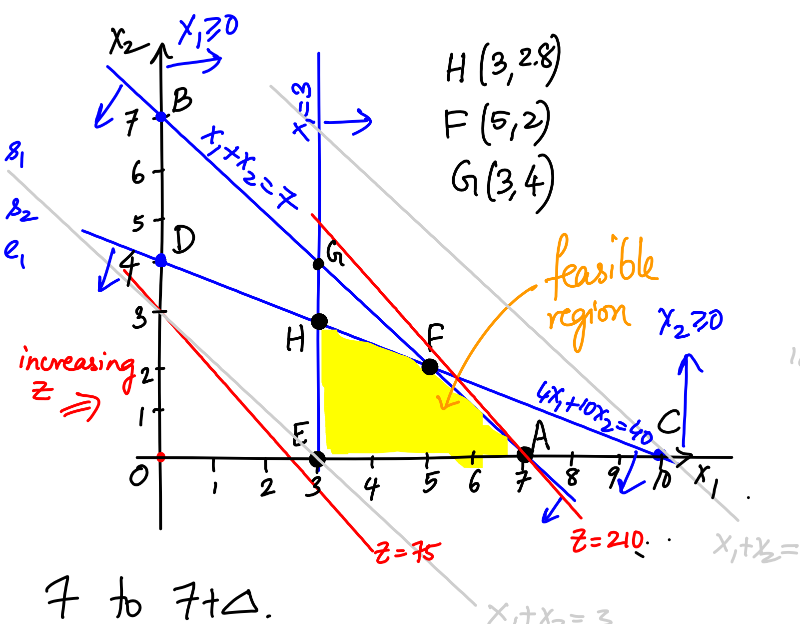
The current basis remains optimal as long as $B^{-1}\bar{b} \geq \bar{0}$ (feasibility).

Note that Row-0 numbers are not affected.

It is helpful to recall the graphical solution:

$$
\begin{array}{llll}
\max & z = 30x_1 + 25x_2 & & \text{(revenue)} \\
\text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land avail.)} \ s_1 \\
& 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \ s_2 \\
& \cancel{10}x_1 & \geq 3\cancel{0} & \text{(min corn)} \ e_1 \\
& x_1, x_2 & \geq 0 & \text{(nonneg)}
\end{array}
$$

Can scale by 10 to get $x_1 \geq 3$.



The optimal solution is
$z^* = 210$.

Consider changing # acres from 7 to $7 + \Delta$.

$$\bar{b} = \begin{bmatrix} 7 \\ 40 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 7+\Delta \\ 40 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 40 \\ 3 \end{bmatrix} + \Delta \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{e}_1\text{, the first unit vector}}$$

More generally, when we change $b_i \to b_i + \Delta$, the new rhs vector is $\bar{b} = \text{old } \bar{b} + \Delta \bar{e}_i$, where $\bar{e}_i$ is the i^{th} unit vector.

$$\bar{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to i$$

New rhs in optimal tableau is given by

$$B^{-1}(\underbrace{\bar{b} + \Delta \bar{e}_i}_{\text{new } \bar{b}}) = \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & a_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \left(\begin{bmatrix} 7 \\ 40 \\ 3 \end{bmatrix} + \Delta \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\bar{e}_1} \right) = \underbrace{B^{-1}\bar{b}}_{\text{original optimal } \bar{x}_B} + \Delta \underbrace{\begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}}_{1^{st}\text{ column of } B^{-1}}$$

$$= \begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \Delta \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4+\Delta \\ 12-4\Delta \\ 7+\Delta \end{bmatrix} \longrightarrow \text{new } \bar{x}_B$$

More generally, new $B^{-1}\bar{b} = \text{old } B^{-1}\bar{b} + \Delta B^{-1}\bar{e}_i$

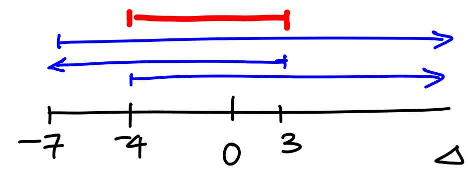
$$= \text{old } B^{-1}\bar{b} + \Delta \underbrace{[B^{-1}]_{\cdot i}}_{i^{th}\text{ column of } B^{-1}}$$

We need the new $\bar{x}_B = \begin{bmatrix} 4+\Delta \\ 12-4\Delta \\ 7+\Delta \end{bmatrix} \geq \bar{0}$ for feasibility, and hence optimality.

$\Rightarrow$ $4+\Delta \geq 0$, $12-4\Delta \geq 0$, and $7+\Delta \geq 0$

$\Rightarrow \Delta \geq -4, \quad \Delta \leq 3, \quad \text{and} \quad \Delta \geq -7.$

$\Longrightarrow \boxed{-4 \leq \Delta \leq 3.}$



As long as there are at least 3 acres ($\Delta = -4$), and at most 10 acres ($\Delta = 3$), we will continue to farm only corn in all of the land. It so happens that in this case, even when $b_1 = 11$, say, i.e., $\Delta = 4$, we would still farm only corn. But the (land available) constraint will no longer be binding, as we have enough labor hours to farm corn in at most 10 acres.

Shadow price of (land) constraint:

$$\text{New objective function value} = \bar{C}_B^T B^{-1}(\text{new } \bar{b}) = \bar{C}_B^T \underbrace{(B^{-1}\text{ new } \bar{b})}_{\text{new } \bar{x}_B}$$

$$= \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 30 \end{bmatrix} \left(\begin{bmatrix} 4 \\ 12 \\ 7 \end{bmatrix} + \Delta \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right)$$

$$\bar{C}_B^T = \begin{array}{ccc} e_3 & s_2 & x_1 \end{array} \\ \begin{bmatrix} 0 & 0 & 30 \end{bmatrix}$$

$$= 210 + 30\Delta$$

shadow price

The shadow price of land constraint = \$30.

Jones would pay upto \$30 for one extra acre of land.

Notice that this price is equal to the revenue from an acre of corn.

If Δ is outside this range, the $\bar{x}_B$ is no longer feasible. The rhs will no longer be ≥ 0, but you can use a dual simplex pivot to reoptimize quickly. More on this topic after we introduce linear programming duality.

Now let's change (# labor hrs) from 40 to $40+\Delta$. Thus, we are changing $b_2 \to b_2+\Delta$, and hence

$$\text{new } \bar{b} = \text{old } \bar{b} + \Delta\begin{bmatrix}0\\1\\0\end{bmatrix} \rightarrow \bar{e}_2 \text{ (2nd unit vector)}$$

$$\Rightarrow \text{New } \bar{x}_B = \text{old } \bar{x}_B + \Delta\,(\text{2nd column of } B^{-1})$$

$$= \begin{bmatrix}4\\12\\7\end{bmatrix} + \Delta\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$= \begin{bmatrix}4\\12+\Delta\\7\end{bmatrix} \geqslant \bar{0} \text{ for feasibility.}$$

$$\Rightarrow 12+\Delta \geqslant 0 \Rightarrow \boxed{\Delta \geqslant -12.}$$

Shadow price:

$$\text{New } z^* = \bar{c}_B^T(\text{new } \bar{x}_B) = [0\ 0\ 30]\left(\begin{bmatrix}4\\12\\7\end{bmatrix} + \Delta\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 210 + (0)\Delta.$$

$\Rightarrow$ Shadow price is zero here, as we are not using all of the 40 hours of labor available (we're using only 28 hrs of labor).

If $\Delta < -12$ here, the current solution becomes infeasible. To get the new optimal solution, we need to do a dual simplex pivot.
(more on this method after we introduce LP duality)

4. Changing the column of a nonbasic variable x_j

Consider changing the revenue/acre of wheat (x_2) from 25 to 35, and at the same time changing the # labor hrs/acre of wheat from 10 to 8. Recall x_2 is non-basic in the optimal tableau. Here is how the column of x_2 in the starting tableau changes:

$$\begin{array}{c} x_2 \\ \hline -25 \\ \hline 1 \\ 10 \\ 0 \\ \hline \end{array} \longrightarrow \begin{array}{c} x_2 \\ \hline -35 \\ \hline 1 \\ 8 \\ 0 \\ \hline \end{array}$$

We can find the column of x_2 in the modified/new optimal tableau directly (using $\bar{c}_B$, B^{-1} from the optimal tableau).

Recall, $\bar{c}_B^T B^{-1} = [30 \quad 0 \quad 0]$ still.

updated column of x_2 in optimal tableau

$$\begin{array}{c} x_j \\ \hline -c_j + \bar{c}_B^T B^{-1} \bar{a}_j \\ \hline B^{-1}\bar{a}_j \\ \hline \end{array} \qquad \begin{array}{c} x_2 \\ \hline -35 + [30 \quad 0 \quad 0]\begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 & -1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 8 \\ 0 \end{bmatrix} \\ \hline \end{array} \qquad \begin{array}{c} x_2 \\ \hline -5 \\ \hline 1 \\ 4 \\ 1 \\ \hline \end{array}$$

new $\bar{a}_2$

Since coefficient of x_2 is Row-0 is not ≥ 0, new tableau is not optimal. But we can reoptimize quickly:

updated column of x_2

	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	0	-5	30	0	0	M	210
e_3	0	0	1	1	0	1	-1	4
s_2	0	0	4	-4	1	0	0	12
x_1	0	1	1	1	0	0	0	7
	1	0	0	25	$5/4$	0	M	225
e_3	0	0	0	2	$-1/4$	1	-1	1
x_2	0	0	1	-1	$1/4$	0	0	3
x_1	0	1	0	2	$-1/4$	0	0	4

New optimal solution is $x_1 = 4$, $x_2 = 3$, $z^* = 225$.

A similar approach can be used when considering a new variable. For instance, Jones could consider growing barley that gives a revenue of \$35/acre and uses 8 hrs/acre of labor. Should he grow any barley? The answer is yes.

$$
\begin{array}{c}
x_3 \\ \hline
-c_3 + \bar{c}_B^T B^{-1} \bar{a}_3 \\ \hline
B^{-1} \bar{a}_3 \\ \hline
\end{array}
\longrightarrow
\begin{array}{c}
x_3 \\ \hline
-5 \\ \hline
1 \\ 4 \\ 1 \\ \hline
\end{array}
$$

using the same calculations done in the previous page.

So we can add this column of x_3 into the original optimal tableau and pivot it in to find the new optimal tableau.