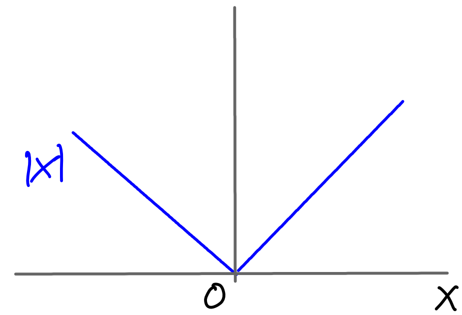
MATH 364: Lecture 21 (10/29/2024)

Today: * LP duality
* motivation for dual LP

Homework 8 Problems



1. The objective function is not linear. $f(x)=|x|$ is a piecewise linear function.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0 \end{cases}$$

Try using idea for modeling urs variables $x \rightarrow x^+, x^-$

Or, try solving two LPs with two (related) objective functions.

2. Consider columns of x_i^+, x_i^- $\quad x_i \leftarrow x_i^+ - x_i^-$

x_i^+	x_i^-
c_i	$-c_i$
a_1	$-a_1$
a_2	$-a_2$
$\vdots$	$\vdots$
a_m	a_m

Show what happen under scaling and replacement EROs $\left(\frac{1}{a_j}\right)R_j$ and $R_k + \alpha R_j$.

Show that the opposite sign property is maintained under both such EROs.

3.

	x_ℓ	x_e
	0	c
	0	a_1
	$\vdots$	a_2
i	1	α
	0	$\vdots$
	0	a_m

what happens after pivot?

Let x_ℓ leave and x_e enter in its place.

* $\alpha > 0$, so that x_e can be pivoted into Row-i.

* $c \leq 0$ to start with.

LP Duality

Associated with every LP (linear program) is another LP called its **dual LP**. The original LP is called the **primal LP**. There are important relationships between the primal and dual LPs, both from the mathematical as well as economic points of view.

A max LP in which every constraint is $\leq$ and all variables are ≥ 0 is a **normal max LP**. Similarly, a min LP in which every constraint is $\geq$ and all variables are ≥ 0 is a **normal min LP**.

A $\geq$ constraint is hence normal for a min-LP, but is opposite to normal for a max LP. Similarly, a $\leq$ constraint is normal for a max-LP, and is opposite to normal for a min-LP.

Nonnegative variables (≥ 0) are always normal (for both max- and min-LPs).

Intuition for normal LPs:
- max revenue s.t. upper bounds on raw materials, i.e., $\leq$ constraints
- min cost s.t. demands (min. requirements), i.e., $\geq$ constraints

Nonnegative variables are always normal.

Let's write the dual LP of a normal max-LP. This dual LP will be a normal min-LP.

Example : Find the dual of the following LP:

(P) for primal

$$\begin{array}{ll} \overset{\text{min}}{\text{max}} & z = 2x_1 + x_2 \\ \text{s.t.} & -x_1 + x_2 \le 1 \quad y_1 \\ & x_1 + x_2 \le 3 \quad y_2 \\ & x_1 - 2x_2 \le 4 \quad y_3 \\ & x_1, x_2 \ge 0 \end{array}$$

(D) for dual

$$\begin{array}{ll} \min & 1y_1 + 3y_2 + 4y_3 \\ \text{s.t.} & -y_1 + y_2 + y_3 \ge 2 \\ & y_1 + y_2 - 2y_3 \ge 1 \\ & y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0 \end{array}$$

Primal (P)		Dual (D)
constraint i ($\le$) normal	$\longleftrightarrow$	variable $y_i \ge 0$ → normal
variable x_j (≥ 0) normal	$\longleftrightarrow$	constraint j ($\ge$) normal
objective : max	$\longleftrightarrow$	objective : min

Dual of a general form normal max-LP

(P)

$$\begin{array}{ll} \max & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \quad y_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \quad y_2 \\ & \vdots \qquad\qquad\qquad \vdots \quad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \quad y_m \\ & x_1, \ x_2, \ldots, \ x_n \ge 0 \end{array}$$

(D)

$$\begin{array}{ll} \min & w = b_1y_1 + b_2y_2 + \cdots + b_my_m \\ \text{s.t.} & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1 \\ & a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2 \\ & \vdots \\ & a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \ge c_n \\ & y_1, \ y_2, \ldots, \ y_m \ge 0 \end{array}$$

"Dual of a dual is Primal": if you take the dual of the dual LP of a given LP, you get the original LP back.

Primal-Dual Relationships

	min LP		max LP	
variables	≥ 0 normal	$\longleftrightarrow$	$\leq$ normal	constraints
	≤ 0 opposite to normal	$\longleftrightarrow$	$\geq$ opposite to normal	
	urs	$\longleftrightarrow$	$=$	
constraints	$\geq$ normal	$\longleftrightarrow$	≥ 0 normal	variables
	$\leq$ opposite to normal	$\longleftrightarrow$	≤ 0 opposite to normal	
	$=$	$\longleftrightarrow$	urs	

In general,

normal variables in (P) $\longleftrightarrow$ normal constraints in (D)

Opposite to normal variables in (P) $\longleftrightarrow$ opposite to normal constraints in (D)

urs variables in (P) $\longleftrightarrow$ $=$ constraints in (D)

You should **not** try to memorize the above table of primal-dual relationships. Instead, use the idea of normal variables/constraints corresponding to normal constraints.

To stress this point, we will rewrite this table in other equivalent forms.

Write the dual LP for the given LPs

1. (P)

$$\begin{array}{llll}
\min & z = x_1 - x_2 & & \\
\text{s.t} & 2x_1 + x_2 \geq 4 & & y_1 \geq 0 \\
 & x_1 + x_2 \geq 1 & & y_2 \geq 0 \\
 & x_1 + 2x_2 \leq 3 & & y_3 \leq 0 \\
 & x_1,\ x_2 \geq 0 & & \\
 & \leq \quad \leq & &
\end{array}$$

(D)

$$\begin{array}{ll}
\max & w = 4y_1 + y_2 + 3y_3 \\
\text{s.t.} & 2y_1 + y_2 + y_3 \leq 1 \\
 & y_1 + y_2 + 2y_3 \leq -1 \\
 & y_1 \geq 0,\ y_2 \geq 0,\ y_3 \leq 0
\end{array}$$

2. (P)

$$\begin{array}{llll}
\min & w = 4y_1 + 2y_2 - y_3 & & \\
\text{s.t.} & y_1 + 2y_2 \leq 6 & & y_1 \leq 0 \\
 & y_1 - y_2 + 2y_3 = 8 & & y_2 \text{ urs} \\
 & y_1,\ y_2 \geq 0,\ y_3 \text{ urs} & & \\
 & \leq \quad \leq \quad = & &
\end{array}$$

could use $\{x_1, x_2\}$ or $\{u_1, u_2\}$ …

(D)

$$\begin{array}{ll}
\max & w = 6y_1 + 8y_2 \\
\text{s.t.} & y_1 + y_2 \leq 4 \\
 & 2y_1 - y_2 \leq 2 \\
 & 2y_2 = -1 \\
 & y_1 \leq 0,\ y_2 \text{ urs}
\end{array}$$

3. (P)

$$\begin{array}{llll}
\max & z = 3x_2 - 4x_1 + 2x_3 & & \\
\text{s.t.} & 2x_1 + 0.5x_3 + 7x_2 \geq 5 & & y_1 \leq 0 \\
 & -3x_2 + 5x_1 \leq -3 & & y_2 \geq 0 \\
 & 2x_1 + 6x_3 = 2 & & y_3 \text{ urs} \\
 & x_4 \geq 5 & & y_4 \leq 0 \\
 & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs},\ x_4 \geq 0 & & \\
 & \geq \quad \leq \quad = \quad \geq & &
\end{array}$$

(D)

$$\begin{array}{ll}
\min & w = 5y_1 - 3y_2 + 2y_3 + 5y_4 \\
\text{s.t.} & 2y_1 + 5y_2 + 2y_3 \geq -4 \\
 & 7y_1 - 3y_2 \leq 3 \\
 & 0.5y_1 + 6y_3 = 2 \\
 & y_4 \geq 0 \\
 & y_1 \leq 0,\ y_2 \geq 0,\ y_3 \text{ urs},\ y_4 \leq 0
\end{array}$$

Notice the variables might not be ordered (or arranged) nicely. But you just have to read down the column of each variable to get the corresponding constraint in the dual.

Motivation behind the dual (how and why)

Farmer Jones LP

max $z = 30x_1 + 100x_2$ (total revenue)

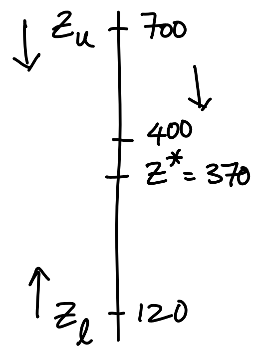
s.t.

$x_1 + x_2 \le 7$ (land)

$4x_1 + 10x_2 \le 40$ (labor hrs)

ignore for now, just for interpretation ~~$10x_1 \ge 30$ (min corn)~~

$x_1, x_2 \ge 0$ (non-neg)



Optimal solution: $x_1 = 3$, $x_2 = 2.8$, $z^* = 370$

Let z_u = upper bound on z^*, z_ℓ = lower bound on z^*.

This is a standard approach to many optimization problems – start with lower and upper bounds for the quantity you are optimizing, and tighten these bounds.

Any feasible point (x_1, x_2) gives a lower bound, e.g., $x_1 = 4, x_2 = 0$, giving $z_\ell = 120$.

we could consider $(7,0)$ or $(4.5, 1.5)$ or $(5,2)$, or ...

How do we get (an) upper bound?

Consider $100 \times$ (land): $100x_1 + 100x_2 \le 700$. But

$$z = 30x_1 + 100x_2 \le 100x_1 + 100x_2 \le 700$$

as long as $x_1, x_2 \ge 0$ (which is true here).

Notice also that the coefficients of x_1, x_2 in z should compare in the right way with the coefficients in 100(land).

The goal is to get smaller and smaller z_u values. May be the (Labor hrs) constraint could give us a smaller z_u value.

$10 \times$ (Labor hrs): $40x_1 + 100x_2 \leq 400$

Again $z = 30x_1 + 100x_2 \leq 40x_1 + 100x_2 \leq 400 \longrightarrow$ new z_u

Let's multiply the (land) and (labor hrs) constraints by y_1 and y_2. We need $y_1 \geq 0$, $y_2 \geq 0$, as the sense of the scaled inequality should stay as $\leq$. We want to say

$$z \leq 7y_1 + 40y_2.$$

But we must be able to compare the coefficients of x_1 and x_2 to those in z properly:

$$
\begin{array}{rl}
y_1(x_1 + x_2 \leq 7) & \text{(land)} \quad + \\
y_2(4x_1 + 10x_2 \leq 40) & \text{(labor hrs)} \\
\hline
\end{array}
$$

$$\underset{\text{IV}}{(y_1 + 4y_2)}\, x_1 + \underset{\text{IV}}{(y_1 + 10y_2)}\, x_2 \leq 7y_1 + 40y_2 = w$$

$$z = 30\,x_1 + 100\,x_2$$

So we need $y_1 + 4y_2 \geq 30$ and $y_1 + 10y_2 \geq 100$. Also, we want to find the smallest upper bound $w = 7y_1 + 40y_2$. Combining all these requirements gives the dual LP!

$$
\begin{array}{lll}
\min & w = 7y_1 + 40y_2 & \\
\text{s.t.} & y_1 + 4y_2 \geq 30 & \quad (D) \\
 & y_1 + 10y_2 \geq 100 & \\
 & y_1,\ y_2 \geq 0 &
\end{array}
$$