# MATH 364: Lecture 22 (10/31/2024)

Today: * economic interpretation of dual LP
* Duality in matrix form – results

## Economic Interpretation of the dual LP for a max LP

Consider the (original) Farmer Jones LP:

$$
\begin{array}{llll}
\max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\
\text{s.t.} & x_1 + x_2 \leq 7 & y_1 \geq 0 & \text{(land)} \\
 & 4x_1 + 10x_2 \leq 40 & y_2 \geq 0 & \text{(labor hrs)} \\
 & \cancel{10x_1 \geq 30} & \cancel{y_3 \leq 0} & \cancel{\text{(min corn)}} \\
 & x_1, x_2 \geq 0 & & \text{(non-neg)}
\end{array}
$$

ignore for now, just for interpretation

$$
\begin{array}{ll}
\min & w = 7y_1 + 40y_2 + 30y_3 \\
\text{s.t.} & y_1 + 4y_2 + 10y_3 \geq 30 \\
 & y_1 + 10y_2 \geq 100 \\
 & y_1 \geq 0, y_2 \geq 0, y_3 \leq 0
\end{array}
$$

We will deal with the (min.corn) constraint, which is opposite to normal, after we explain the rest of the problem.

Suppose a firm wants to buy the farming enterprise from Jones. The firm needs to make an offer, i.e., unit price, for every acre and every labor hour Jones has. The firm would like to buy Jones' enterprise at minimum cost. Thus, the firm quotes prices $y_1$ and $y_2$ for each acre of land and hour of labor, respectively. The total cost for the firm is hence $w = 7y_1 + 40y_2$, and it tries to minimize $w$.

Hence its objective function is

$$\min \quad w = 7y_1 + 40y_2 \qquad \text{(cost)}$$

At the same time, the offer should be attractive to Jones.

If Jones has 1 acre of land and 4 hrs of labor, he can farm corn in that acre and make \$30 revenue.
Hence the prices the firm offers should be such that they match this revenue, i.e.,

$$y_1 + 4y_2 \geq 30 \quad \text{(match revenue from corn)}$$

Similarly, for wheat, we should have

$$y_1 + 10y_2 \geq 100 \quad \text{(match revenue from wheat)}$$

$y_1, y_2$ are unit prices quoted by the firm, so should be non negative.

Putting it all together, we get the dual LP, which captures the problem from the competing firm's perspective.

$$
\begin{array}{lll}
\min & w = 7y_1 + 40y_2 & \text{(total cost)} \\
\text{s.t.} & y_1 + 4y_2 \geq 30 & \text{(match revenue from corn)} \\
 & y_1 + 10y_2 \geq 100 & \text{(match revenue from wheat)} \\
 & y_1, y_2 \geq 0 & \text{(non-neg)}
\end{array}
$$

What about (min-corn) constraint?

We first modify the dual LP to include $y_3$:

$$\begin{aligned}
\min\ & w = 7y_1 + 40y_2 + 30y_3 \\
\text{s.t.}\ & y_1 + 4y_2 + 10y_3 \geq 30 \\
& y_1 + 10y_2 \geq 100 \\
& y_1 \geq 0,\ y_2 \geq 0,\ y_3 \leq 0
\end{aligned}$$

(D) with $y_3$ included

Jones was making 30 bushels of corn/week. Hence the firm could sell off those 30 (or more) bushels of corn at a price of $-y_3$, and make some revenue that offsets its total cost.

$-y_3$ because it is in the reverse sense of $y_1$ and $y_2$

Hence, it will sell each bushel of corn at $-y_3$ dollars, such that $y_3 \leq 0$.

# Economic Interpretation of the dual of a (normal) min-LP

## Gaseous Chemicals LP (from HW2)

3. (25) Gaseous Chemicals makes three chemicals A, B, and C, via two processes. Running Process 1 for an hour costs \$4, and yields 3, 1, and 1 units of A, B, and C, respectively. Running Process 2 for an hour costs \$1, and yields 1 unit of A and 1 unit of B. At least 10, 5, and 3 units of A, B, and C, respectively, must be produced in order to meet demand. Determine the daily production plan that minimizes the total daily cost for meeting the demands of Gaseous Chemicals using the graphical method to solve LPs.

(P)

$$
\begin{array}{llrcll}
\min & z = & 4x_1 + x_2 & & & \text{(total cost)} \\
\text{s.t.} & & 3x_1 + x_2 & \geq & 10 & \text{(demand A)} \quad y_1 \geq 0 \\
 & & x_1 + x_2 & \geq & 5 & \text{(demand B)} \quad y_2 \geq 0 \\
 & & x_1 & \geq & 3 & \text{(demand C)} \quad y_3 \geq 0 \\
 & & x_1,\ x_2 & \geq & 0 & \text{(non-negativity)} \\
 & & \leq\ \leq & & &
\end{array}
$$

(D)

$$
\begin{array}{ll}
\max & w = 10y_1 + 5y_2 + 3y_3 \\
\text{s.t.} & 3y_1 + y_2 + y_3 \leq 4 \\
 & y_1 + y_2 \leq 1 \\
 & y_1,\ y_2,\ y_3 \geq 0
\end{array}
$$

Suppose a firm wants to sell chemicals A, B, C to Gaseous. The firm quotes unit prices $y_1, y_2, y_3$ for A, B, C. The firm tries to maximize its revenue: $w = 10y_1 + 5y_2 + 3y_3$. Gaseous would not buy more than 10 units of A, which is the demand for A, and similarly for B and C.

The idea here is that Gaseous could buy the finished products (chemicals A, B, C) from the other firm to meet the corresponding demands, rather than make the products themselves by running processes 1 and 2.

The offer should be attractive to Gaseous. If Gaseous has \$4, they could run Process 1 for 1 hour and make 3 units of A, 1 unit of B, and 1 unit of C. Hence the total amount Gaseous has to pay for 3,1,1 units of A,B,C, respectively cannot be more than \$4. Hence

$$3y_1 + y_2 + y_3 \le 4 \quad \text{(match Process 1 cost)}$$

Similarly, for Process 2, we get $y_1 + y_2 \le 1$
(match Pr 2 cost)

$y_1, y_2, y_3$ are unit prices, so should be $\geq 0$.

If there is an opposite-to-normal constraint, i.e., a $\le$ constraint, in the primal min-LP, you could interpret it in a way similar to how we interpreted the (min-corn) constraint in the Farmer Jones LP. The second firm would have to pay for this raw material here, for instance, and would quote a unit price for the same, which would be $\le 0$ as compared to $y_1, y_2, y_3$ here.

# Duality in Matrix Form

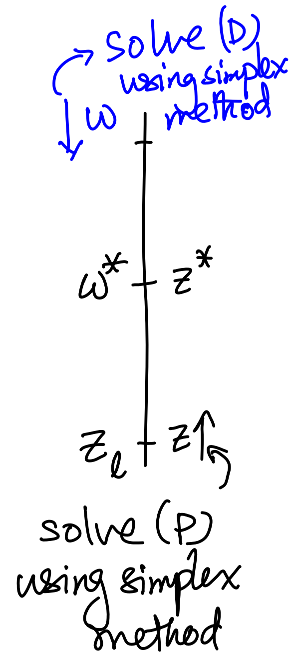
→ to explore the connections between the primal and dual LPs in depth, we switch to matrix notation now.

(P)
$$\max\ z = \bar{c}^T\bar{x}$$
$$\text{s.t.}\quad A\bar{x} \le \bar{b} \qquad \bar{y} \ge \bar{0}\ \text{(m-vector)}$$
$$\bar{x} \ge \bar{0}$$

normal max-LP

$$\min\ w = \bar{b}^T\bar{y}$$
$$A^T\bar{y} \ge \bar{c} \qquad (D)$$
$$\bar{y} \ge \bar{0}$$

normal min-LP




weak because it specifies a '≤' result as opposed to an '=' result

**Lemma 1** (weak duality) If $\bar{x}$ is feasible for (P) and $\bar{y}$ is feasible for (D), we have $z = \bar{c}^T\bar{x} \le \bar{b}^T\bar{y} = w$.

**Proof** $\bar{x}$ is feasible for (P) $\Rightarrow$ $A\bar{x} \le \bar{b}$, $\bar{x} \ge \bar{0}$

$\bar{y}$ is feasible for (D) $\Rightarrow$ $A^T\bar{y} \ge \bar{c}$, $\bar{y} \ge \bar{0}$

$$\bar{y}^T(A\bar{x} \le \bar{b}) \Rightarrow \bar{y}^TA\bar{x} \le \bar{y}^T\bar{b} = \bar{b}^T\bar{y} = w.$$

$$(A^T\bar{y} \ge \bar{c})^T \Rightarrow (\bar{y}^TA \ge \bar{c}^T)\bar{x} \Rightarrow \bar{y}^TA\bar{x} \ge \bar{c}^T\bar{x} = z.$$

Combining, we get $z = \bar{c}^T\bar{x} \le \bar{y}^TA\bar{x} \le \bar{b}^T\bar{y} = w$. $\square$

denotes "end of proof" also "Q.E.D."

The result (of weak duality) holds for general primal-dual LP pairs, not just for normal LPs.

Exploiting the primal-dual relationships is a standard practice in solving most optimization problems. Hence, we could solve (P) to optimality, or, alternatively, we could solve (D) to optimality. As the next Lemma states, solving one of them to optimality is guaranteed to equivalently solve the other problem as well.

But another versatile idea is to combine the solution process for both (P) and (D). Thus, one could switch back and forth between (P) and (D), and tighten both bounds simultaneously. Such methods are called primal-dual algorithms.

**Lemma 2** (Strong duality) If $z = \bar{c}^T\bar{x} = \bar{b}^T\bar{y} = w$ for $\bar{x}, \bar{y}$ feasible for (P) and (D), respectively, then $\bar{x}, \bar{y}$ are optimal for (P) and (D), respectively.

**Proof** All $z$ values lie below all $w$ values (Lemma 1). Hence when $z = w$, we get optimality for both.