MATH 364: Lecture 23 (11/05/2024)

Today:
* dual theorem
* reading off optimal $\bar{y}$ from primal tableau
* shadow price$_i$ = y_i

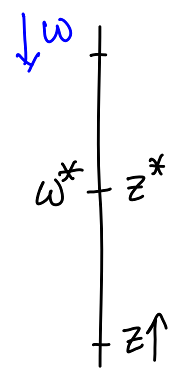
We first present two more results that connect unboundedness of an LP with the infeasibility of its dual LP.

Recall: Lemma 1 (weak duality) $z = \bar{c}^T\bar{x} \le \bar{b}^T\bar{y} = w$ for $\bar{x} \in (P)$, $\bar{y} \in (D)$

Lemma 2 (strong duality) If $z = \bar{c}^T\bar{x} = \bar{b}^T\bar{y} = w$ then $\bar{x}, \bar{y}$ are optimal for (P) and (D), respectively.

Lemmas 3 and 4 If (P) is unbounded, then (D) is infeasible. Similarly, if (D) is unbounded, then (P) is infeasible.

If (P) is unbounded, we can push z up without limits. Hence there are no finite w values, i.e., there are no feasible $\bar{y}$ for (D).



Note: (P) infeasible does not imply that (D) is unbounded.

We can create instances where both (P) and (D) are infeasible — see below. Here, both (P) and (D) are obviously infeasible.

$$\begin{array}{llll} & \max\ z = x_1 + 2x_2 & & \\ \text{s.t.} & x_1 + x_2 = 1 & y_1 \text{ urs} \\ (P) & 2x_1 + 2x_2 = 3 & y_2 \text{ urs} \\ & x_1, x_2 \text{ urs} & \end{array}$$

$$\begin{array}{lll} \min\ w = y_1 + 3y_2 & & \\ \text{s.t.} & y_1 + 2y_2 = 1 & (D) \\ & y_1 + 2y_2 = 2 & \\ & y_1, y_2 \text{ urs} & \end{array}$$

The Dual Theorem

Let $\bar{x}_B$ be the optimal basic solution to (P), B be the basis matrix, $\bar{c}_B$ the basic cost vector. Then $\bar{y}^T = \bar{c}_B^T B^{-1}$ is optimal for (D), and $z^* = w^* = \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}^T} \bar{b} = \bar{y}^T \bar{b}$.

While we were working with (P) as a normal max-LP and hence (D) as a normal min-LP, this result holds even when (P) is a general max-LP.

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	$-\bar{c}_B^T$	$-\bar{c}_N^T$	0
0	B	N	$\bar{b}$

z	$\bar{x}_B$	$\bar{x}_N$	rhs
1	0	$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N$	$\bar{c}_B^T B^{-1} \bar{b}$
0	I	$B^{-1} N$	$B^{-1} \bar{b}$

IDEA Start with $\bar{y}^T = \bar{c}_B^T B^{-1}$.

Check that $\bar{y}$ is feasible for (D), and then check that $w = \bar{y}^T \bar{b} = z$

Optimality of the tableau for (P): $-\bar{c}^T + \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}^T} A \geq \bar{0}$

(all numbers under $\bar{x}$ in Row-0, or z-Row, are ≥ 0 for optimality)

Setting $\bar{y}^T = \bar{c}_B^T B^{-1} \Rightarrow -\bar{c}^T + \bar{y}^T A \geq \bar{0} \Rightarrow A^T \bar{y} \geq \bar{c}$

Consider the optimality criteria for slack variables:

$$\bar{0}^T + \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}^T}(I) \geq \bar{0}^T \Rightarrow \bar{y}^T \geq \bar{0}^T \Rightarrow \bar{y} \geq \bar{0}$$

Hence $\bar{y}^T = \bar{c}_B^T B^{-1}$ is feasible for (D).

But $w^* = \bar{b}^T \bar{y} = \bar{y}^T \bar{b} = \bar{c}_B^T B^{-1} \bar{b} = z^*$ in the optimal primal tableau. Hence by Lemma 2 (strong duality), $\bar{y}$ is optimal for (D).

<u>Implication</u> We can read off the optimal dual solution from the optimal primal tableau.

The Row-0 (or z-Row) in the optimal tableau is

$-\bar{c}^T + \bar{c}_B^T B^{-1} A$ (in general),

$-\bar{c}_B^T + \bar{c}_B^T B^{-1} B = \bar{0}$ for $\bar{x}_B$,

$-\bar{c}_N^T + \bar{c}_B^T B^{-1} N$ for $\bar{x}_N$, and in particular,

$\bar{0} + \underbrace{\bar{c}_B^T B^{-1}}_{\bar{y}} I$ for slack variables

Hence we can read off the optimal $\bar{y}$ under slack columns for a normal max-LP, and more generally as follows.

Expressions in Row-0:

$\bar{0} + \bar{c}_B^T B^{-1} I$ for slack variables

$\bar{0} + \bar{c}_B^T B^{-1}(-I)$ for excess variables,

$M + \bar{c}_B^T B^{-1}(I)$ for artificial variables.

Hence, the optimal value of y_i (dual variable for constraint i) in a max-LP:

constraint i is $\leq$: coefficient of s_i in Row-0

constraint i is $\geq$: $-$(coefficient of e_i in Row-0)

constraint i is $=$: (coefficient of a_i in Row-0) $- M$

Illustration

$$\begin{aligned} \max \quad & 30x_1 + 25x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 7 \quad y_1 \ge 0 \\ & 4x_1 + 10x_2 \le 40 \quad y_2 \ge 0 \\ & 10x_1 \ge 30 \quad y_3 \le 0 \\ & x_1, x_2 \ge 0 \\ & \ \ \ge \ \ \ge \end{aligned}$$

$$\begin{aligned} \min \ w = \ & 7y_1 + 40y_2 + 30y_3 \\ \text{s.t.} \quad & y_1 + 4y_2 + 10y_3 \ge 30 \\ (D) \quad & y_1 + 10y_2 \ge 25 \\ & y_1, y_2 \ge 0, \quad y_3 \le 0 \end{aligned}$$

Optimal tableau (from Lecture 19):

BV	z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
	1	0	5	30	0	0	M	210
e_3	0	0	1	1	0	1	−1	4
s_2	0	0	6	−4	1	0	0	12
x_1	0	1	1	1	0	0	0	7

From the optimal tableau, $y_1 = 30$, $y_2 = 0$, $y_3 = 0$ is optimal for (D).
Note that $w = 7 \times 30 = 210 = z^*$, and hence must indeed be optimal.

Notice how we could read off the y_3 value from under either the e_3 or a_3 column here.

$$y_3 = -(\text{coefficient in Row-0 under } e_3) = 0$$

$$y_3 = (\text{Coefficient in Row-0 under } a_3) - M = M - M = 0.$$

Illustration on Farmer Jones LP

$$\max\ z = 30x_1 + 100x_2$$
s.t.
$$x_1 + x_2 \le 7 \quad y_1$$
$$4x_1 + 10x_2 \le 40 \quad y_2$$
$$10x_1 \ge 30 \quad y_3$$
$$x_1, x_2 \ge 0$$

$$\min\ w = 7y_1 + 40y_2 + 30y_3$$
s.t.
$$y_1 + 4y_2 + 10y_3 \ge 30 \qquad (D)$$
$$y_1 + 10y_2 \ge 100$$
$$y_1 \ge 0,\ y_2 \ge 0,\ y_3 \le 0$$

$x_1 = 3, x_2 = 2.8, s_1 = 1.2$ with $z^* = 370$.

See the Matlab session on the course web page. We use $BV = \{x_1, x_2, s_1\}$ to directly compute the optimal tableau.

Here is the optimal tableau:

$M-1$ // 10,000

z	x_1	x_2	s_1	s_2	e_3	a_3	rhs
1	0	0	0	10	1	9999	370
0	1	0	0	0	-1/10	1/10	3
0	0	1	0	1/10	1/25	-1/25	14/5 = 2.8
0	0	0	1	-1/10	3/50	-3/50	6/5 = 1.2

$y_1 = (\text{coefficient of } s_1 \text{ in Row-0}) = 0.$

$y_2 = (\text{coefficient of } s_2 \text{ in Row-0}) = 10.$

$y_3 = -(\text{coefficient of } e_3 \text{ in Row-0}) = -10.$

Also, $y_3 = (\text{coefficient of } a_3 \text{ in Row-0}) - M = 9999 - 10{,}000 = -1.$

Shadow Price of constraint $i = y_i$ (optimal value of dual variable)

Change $b_i \leftarrow b_i + \Delta$, find new optimal solution $(\bar{x}_B^\Delta)$ assuming the basis remains same. Then find new optimal z^*, z^*_Δ, and write $z^*_\Delta = z^* + p_i \Delta$. Then p_i is the shadow price.

By strong duality, $z^* = w^*$, and $z^*_\Delta = w^*_\Delta$.

w^*_Δ → optimal dual objective function value with $b_i + \Delta$

But $w^* = b_1 y_1 + \cdots b_i y_i + \cdots + b_m y_m$ and $w^*_\Delta = b_1 y_1 + \cdots + (b_i + \Delta) y_i + \cdots + b_m y_m$

$$\Rightarrow \quad w^*_\Delta = \underbrace{b_1 y_1 + \cdots + b_i y_i + \cdots + b_m y_m}_{w^*} + y_i \Delta = w^* + y_i \Delta$$

$$\Rightarrow \quad z^*_\Delta = w^*_\Delta = w^* + y_i \Delta = z^* + y_i \Delta.$$

Hence $p_i = y_i$, i.e., shadow price of constraint $i = y_i$.

Here, $y_1 = 0$, $y_2 = 10$, $y_3 = -1$. So, shadow price of (land) $= 0$, that of (labor hours) $= 10$, and that of (min-corn) $= -1$.

If the min-corn requirement goes up by 1 unit, i.e., from 30 to 31, the rhs as written here would go up from 30 to 31, and the z^* value will decrease from 370 to $370 + (-1) \cdot 1 = 369$.

(See the course web page for the AMPL session).