MATH 364 : Lecture 24 (11/07/2024)

Today: * complementary slackness conditions (CSC)

Complementary Slackness Conditions (CSCs)

$$(P) \quad \begin{array}{ll} \max & \bar{c}^T\bar{x} \\ \text{s.t.} & A\bar{x} \le \bar{b} \quad \bar{y} \geqslant \bar{0} \\ & \bar{x} \geqslant \bar{0} \\ & \quad \geqslant \end{array} \qquad\qquad \begin{array}{ll} \min & w = \bar{b}^T\bar{y} \\ \text{s.t.} & A^T\bar{y} \geqslant \bar{c} \\ & \bar{y} \geqslant \bar{0} \end{array} \quad (D)$$

Let $\bar{x}$ and $\bar{y}$ be feasible for (P) and (D), respectively.

Optimality conditions:

- $-\bar{c}^T + \bar{c}_B^T B^{-1} A \geqslant \bar{0}^T$ (Row-0 in tableau simplex)
- If $z = \bar{c}^T\bar{x} = \bar{b}^T\bar{y} = w$, then $\bar{x}$ and $\bar{y}$ are optimal for (P) and (D), respectively
- CSCs

Naturally, all three optimality conditions are equivalent.

CSCs

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

To convert (P) and (D) to standard form, we use slack variables $s_1, \dots, s_m$ in (P), and excess variables $e_1, e_2, \dots, e_n$ in (D). Equivalently, let

$$\bar{s} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{bmatrix} \quad \text{and} \quad \bar{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

<u>CSCs</u> $\bar{x}$ and $\bar{y}$ are optimal for (P) and (D), respectively if and only if

$$s_i y_i = 0 \quad \text{for } i=1,\dots,m$$

$$\text{and} \quad e_j x_j = 0 \quad \text{for } j=1,\dots,n$$

In words, the product of slack/excess variable and the corresponding dual variable is zero at optimality. Hence, at least one of them is zero!

Equivalently, if a constraint is non-binding, the corresponding variable in the complementary (i.e., dual) problem must be zero at optimality.

Recall: Farmer Jones LP:

$$\begin{aligned} \max\ & z = 30x_1 + 100x_2 \\ \text{s.t.}\ & x_1 + x_2 \le 7 \quad y_1 \\ & 4x_1 + 10x_2 \le 40 \quad y_2 \\ & x_1 \ge 3 \quad y_3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} \min\ & w = 7y_1 + 40y_2 + 3y_3 \\ \text{s.t.}\ & y_1 + 4y_2 + y_3 \ge 30 \quad e_1 \\ & y_1 + 10y_2 \ge 100 \quad e_2 \\ & y_1 \ge 0,\ y_2 \ge 0,\ y_3 \le 0 \end{aligned}$$

$x_1 = 3,\ x_2 = 2.8,\ z^* = 370$

$s_1 = 1.2,\ s_2 = 0,\ e_3 = 0$

$\underset{0}{s_1 y_1} = 0,\ \underset{0}{s_2 y_2} = 0,\ \underset{0}{e_3 y_3} = 0$ ✓

$y_1 = 0,\ y_2 = 10,\ y_3 = -10,\ w^* = 370$

$e_1 = 0,\ e_2 = 0$

$\underset{0}{e_1 x_1} = 0,\ \underset{0}{e_2 x_2} = 0$ ✓

CSCs hold for any pair of (P)/(D) LPs, not just for normal LPs.

Using Complementary Slackness

For the given LP, solve its dual LP, and then use CSCs to solve the original LP.

(P)

$$\max\ z = 5x_1 + 3x_2 + x_3$$

s.t.

$$2x_1 + x_2 + x_3 \le 6 \quad y_1 \ge 0 \quad s_1$$

$$x_1 + 2x_2 + x_3 \le 7 \quad y_2 \ge 0 \quad s_2$$

$$x_1, x_2, x_3 \ge 0$$

$$\ge \quad \ge \quad \ge$$

(D)

$$\min\ w = 6y_1 + 7y_2$$

s.t.

$$2y_1 + y_2 \ge 5 \quad e_1$$

$$y_1 + 2y_2 \ge 3 \quad e_2$$

$$y_1 + y_2 \ge 1 \quad e_3$$

$$y_1, y_2 \ge 0$$

Optimal solution (D): $y_1 = \frac{7}{3}$. $y_2 = \frac{1}{3}$, $w^* = \frac{49}{3}$.

In (D), constraint 1: $2\left(\frac{7}{3}\right) + \frac{1}{3} = 5 \Rightarrow e_1 = 0$

$$\left(\frac{7}{3}\right) + 2\left(\frac{1}{3}\right) = 3 \Rightarrow e_2 = 0$$

$$\left(\frac{7}{3}\right) + \left(\frac{1}{3}\right) = 8/3 \Rightarrow e_3 = 5/3$$

By CSCs, $x_3 = 0$ at optimality, as $e_3 x_3 = 0$.
Also, as $y_1 > 0$ and $y_2 > 0$, we get $s_1 = 0$, $s_2 = 0$ (as $s_i y_i = 0$).

$$\left.\begin{aligned} 2x_1 + x_2 + \underbrace{\cancel{x_3}}_{=0} &\cancel{\le}\ 6 \\ x_1 + 2x_2 + \underbrace{\cancel{x_3}}_{=0} &\cancel{\le}\ 7 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2x_1 + x_2 &= 6 \\ x_1 + 2x_2 &= 7 \end{aligned}\right\} \quad x_2 = 8/3,\ x_1 = 5/3$$

So $x_1 = 5/3$, $x_2 = 8/3$ is optimal.

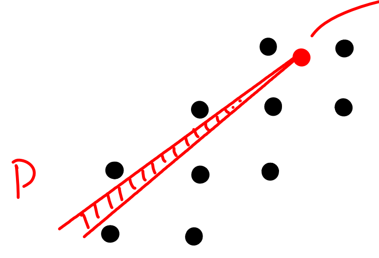
Indeed, $z^* = 5\left(5/3\right) + 3\left(8/3\right) = \frac{49}{3} = w^*$.

Of course, the use of CSCs is more widespread than indicated by the above toy example. There are classes of optimization algorithms based on each type of optimality conditions. The ones based on CSCs start with pairs of solutions $\bar{x}$ and $\bar{y}$ that do not satisfy all CSCs, but may be satisfy feasibility for (P) and (D), and then progressively satisfy the CSCs. The economic interpretation is also quite important.

In the next lecture, we will talk about integer programming!

Q: Could we just round the continuous solution?

Might not even be feasible!!



rounding this corner point in any fashion puts us out of the feasible region.