MATH 364: Lecture 26 (11/14/2024)

Today: * Fixed charge IP
* either-or statements

2. Fixed charge (or setup cost) problem

3 A manufacturer can sell product 1 at a profit of \$2/unit and product 2 at a profit of \$5/unit. Three units of raw material are needed to manufacture 1 unit of product 1, and 6 units of raw material are needed to manufacture 1 unit of product 2. A total of 120 units of raw material are available. If any of product 1 is produced, a setup cost of \$10 is incurred, and if any of product 2 is produced, a setup cost of \$20 is incurred. Formulate an IP to maximize profits.

Decisions: ① produce any of product 1,2 at all? YES/NO
② how many of each to produce?

d.v.'s

Let $y_i = \begin{cases} 1 & \text{if we make any of product } i \\ 0, & \text{otherwise} \end{cases}, \quad i=1,2$

and x_i = # units of product i made, $i=1,2$.

"element of"

We need $y_i \in \{0,1\}$, $x_i \geq 0$

So, $y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{otherwise} \end{cases}, \quad i=1,2.$ This is just the definition (or description) of y_i.

We need to enforce the relationship between y_i and x_i using linear inequalities.

Constraints

$$3x_1 + 6x_2 \leq 120 \quad \text{(raw matl.)}$$

$$x_1 \leq M_1 y_1, \quad M_1 > 0$$
$$x_2 \leq M_2 y_2, \quad M_2 > 0$$
(M_i large, positive).

$M_1 = \frac{120}{3} = 40$ and $M_2 = \frac{120}{6} = 20$ work here.

If no such info is known, can use $M_1 = M_2 = 10^5$, say.

Let's see why these constraints are correct:

$$x_1 \le M_1 y_1$$

If $x_1 > 0$, the only way this constraint will hold is with $y_1 = 1$.
Hence $y_1 = 1$ when $x_1 > 0$.
If $x_1 = 0$, the constraint holds with $y_1 = 0$ or with $y_1 = 1$.
In particular, $x_1 = 0$ does not force $y_1 = 0$ here. We will have the objective function doing this forcing!

Objective function: $\max z = \underbrace{2x_1 + 5x_2}_{\text{revenue}} - \underbrace{10y_1 - 20y_2}_{\text{costs}}$ (profit)

The coefficient of y_1 is -10 in a max obj. fn, hence y_1 is forced to 0 when possible. Hence when $x_1 = 0$, we get $y_1 = 0$ in the optimal solution.

The whole MIP:

$$\begin{aligned}
\max\ z = 2x_1 + 5x_2 - 10y_1 - 20y_2 & \qquad \text{(profit)} \\
\text{s.t.} \quad 3x_1 + 6x_2 &\le 120 \quad \text{(raw matl.)} \\
x_1 &\le 40y_1 \quad \text{(forcing const 1)} \\
x_2 &\le 20y_2 \quad (\text{" \quad " } 2) \\
x_1, x_2 \ge 0,\ & y_1, y_2 \in \{0,1\}.
\end{aligned}$$

$x_1 \le M_1 y_1$: With $y_1 = 1$, this constraint specifies an upper bound on x_1; hence we use the bound as suggested by raw material availability.

In general, we write $\sum_j \alpha_j x_j \le M y_i$. $\xrightarrow{} 0,1$

When $y_i = 1$, the constraint is $\sum_j \alpha_j x_j \le M$.

We use the smallest M that makes sense from data. If not able to estimate, use $M = 10^5$, say, or some similar large number.

See the course web page for AMPL files.

When should we insist on integrality?

1. Should we build dorm on campus? YES/NO

2. How many rooms to include?

We do need binary variable here...

Could get away w/ a continuous variable.

say, $x = 234.6$; Choosing $x = 234$ or 235 may not make a huge difference.

Either-Or constraint

$$f(x_1,\dots,x_n) \le 0 \quad \text{——— (1)}$$
$$g(x_1,\dots,x_n) \le 0 \quad \text{——— (2)}$$

Model: Either (1) or (2) must hold.

Use an extra binary var:

Let $y = \begin{cases} 1 & \text{if (2) holds, and} \\ 0 & \text{otherwise} \Rightarrow \text{(1) holds} \end{cases}$

Assume there is $M > 0$ large enough such that $f(x_1,\dots,x_n) \le M$ and $g(x_1,\dots,x_n) \le M$ always hold.

Model:

$$\boxed{\begin{aligned} f(x_1,\dots,x_n) &\le My && \text{——— (3)} \\ g(x_1,\dots,x_n) &\le M(1-y) && \text{——— (4)} \\ y &\in \{0,1\} && \text{——— (5)} \end{aligned}}$$

If $y=0$, (3) $\Rightarrow \underbrace{f(\cdot) \le 0}_{(1)}$ and (4) $\Rightarrow \underbrace{g(\cdot) \le M}_{\text{redundant}}$

If $y=1$ (3) $\Rightarrow \underbrace{f(\cdot) \le M}_{\text{redundant}}$ and (4) $\Rightarrow \underbrace{g(\cdot) \le 0}_{(2)}$.

Notice $f(\cdot) \le M$ is always true, and is **not** implying (2) holds.

In the basketball starting line-up problem, we had

(5) either player 2 or 3 must start.

We wrote $x_2 + x_3 \geq 1$ ← does allow both to start (from logic).

$x_2 = 1$ $\quad$ $x_3 = 1$

or $x_2 \geq 1$, $x_3 \geq 1$

So, we want $1 - x_2 \leq 0$

or $1 - x_3 \leq 0$

$$\boxed{\begin{array}{l} 1 - x_2 \leq y \\ 1 - x_3 \leq 1 - y \\ y \in \{0,1\} \end{array}}$$

M=1 works here. So we can write

$1 - x_2 \leq 1$ as $x_2 \in \{0,1\}$

When $y = 1$, we get $1 - x_2 \leq 1 \Rightarrow x_2 \geq 0$ ✓

$1 - x_3 \leq 0 \Rightarrow x_3 \geq 1$

Adding these two inequalities, we get $\cancel{1} - x_2 + 1 - x_3 \leq \cancel{1}$

i.e., $x_2 + x_3 \geq 1$

If we want to allow both (1) and (2) to hold together, we can write

$$f(\cdot) \leq M(1 - y_f)$$
$$g(\cdot) \leq M(1 - y_g)$$
$$y_f + y_g \geq 1$$
$$y_f, y_g \in \{0,1\}$$

Note: If we write $y_f + y_g = 1$, we get the previous model.

If we have more than two alternatives:

$$\left.\begin{aligned} f_1(\cdot) &\le 0 \\ &\vdots \\ f_k(\cdot) &\le 0 \end{aligned}\right\}$$ at least one alternative holds

We can write

$$\begin{aligned} f_1(\cdot) &\le M(1-y_1) \\ &\vdots \\ f_k(\cdot) &\le M(1-y_k) \\ y_1 + y_2 + \cdots + y_k &\geq 1 \\ y_i &\in \{0,1\} \end{aligned}$$