MATH 364: Lecture 28 (11/21/2024)

Today: * IP formulations

1. Model: $x=1$ or $x=2$ or $x=3$ or $x=4$

Here: ($1 \leq x \leq 4$, x integer) works. But what if the options were 15, 7, 23, 48 ?

Original statement is

$x \leq 1$ and $x \geq 1$ OR $x \leq 2$ and $x \geq 2$ OR $x \leq 3$ and $x \geq 3$ OR $x \leq 4$ and $x \geq 4$

let $t_1 \in \{0,1\}$ $\quad t_2 \in \{0,1\}$ $\quad t_3 \in \{0,1\}$ $\quad t_4 \in \{0,1\}$

$x-1 \leq 0$ OR $x-2 \leq 0$ OR $x-3 \leq 0$ OR $x-4 \leq 0$

$$
\begin{aligned}
x-1 &\leq M(1-t_1) \\
x-2 &\leq M(1-t_2) \\
x-3 &\leq M(1-t_3) \\
x-4 &\leq M(1-t_4)
\end{aligned}
$$

$M=4$ works here

$t_1+t_2+t_3+t_4=1$

$t_i \in \{0,1\}, \; i=1,2,3,4$

The second set of four alternatives can be modeled as follows:

$$-x+1 \le M(1-t_1)$$
$$-x+2 \le M(1-t_2)$$
$$-x+3 \le M(1-t_3)$$
$$-x+4 \le M(1-t_4)$$

We already have

$$t_1+t_2+t_3+t_4=1$$
$$t_i \in \{0,1\},\ i=1,2,3,4.$$

Can put them all together:

$$x-1 \le M(1-t_1) \qquad -x+1 \le M(1-t_1)$$
$$x-2 \le M(1-t_2) \qquad -x+2 \le M(1-t_2)$$
$$x-3 \le M(1-t_3) \qquad -x+3 \le M(1-t_3)$$
$$x-4 \le M(1-t_4) \qquad -x+4 \le M(1-t_4)$$

$$t_1+t_2+t_3+t_4 \overset{\geq}{=} 1$$
$$t_i \in \{0,1\},\ i=1,2,3,4.$$

→ would also work here → just that we'll never have more than one $t_i=1$ at the same time

In general, if you're not sure whether it is an XOR or inclusive OR, you could go either way.

2. Model the following statement using extra binary variables:
if $x \le 2$ then $y \le 3$, where $x, y \in \mathbb{Z}$ (integers)

$A \Rightarrow B \equiv \text{not} A$ OR B

We want alternatives expressed in $f(\cdot) \le 0$, $g(\cdot) \le 0$, etc. form.

Statement is equivalent to

either $x > 2$ or $y \le 3$

$\equiv$ either $x \ge 3$ or $y \le 3$, as $x \in \mathbb{Z}$

$\equiv$ either $-x + 3 \le 0$ or $y - 3 \le 0$

$$\boxed{\begin{aligned} -x+3 &\le Mt \\ y-3 &\le M(1-t) \\ t &\in \{0,1\} \end{aligned}}$$

XOR!

We cannot estimate a value for M here — just leave as M.

$$\boxed{\begin{aligned} -x+3 &\le M(1-t_1) \\ y-3 &\le M(1-t_2) \\ t_1+t_2 &\ge 1 \\ t_1, t_2 &\in \{0,1\} \end{aligned}}$$

This formulation allows either one or both alternatives to hold.

3. Model: if $x+2y>2$ holds, then either $2x+3y\le 5$ holds or $3x+4y\ge 4$ holds.

(Note: x, y are not assumed to be integers here.)

Statement is equivalent to

either $x+2y\le 2$ or $(2x+3y\le 5$ or $3x+4y\ge 4)$

$\equiv \quad x+2y-2\le 0$ or $2x+3y-5\le 0$ or $-3x-4y+4\le 0$

$$\boxed{\begin{aligned} x+2y-2 &\le M(1-t_1) \\ 2x+3y-5 &\le M(1-t_2) \\ -3x-4y+4 &\le M(1-t_3) \\ t_1+t_2+t_3 &\ge 1 \\ t_1, t_2, t_3 &\in \{0,1\} \end{aligned}}$$

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4. Model: if $|x| \leq 4$ then $|y| > 5$, where $x, y \in \mathbb{Z}$.

Statement is equivalent to

either $|x| > 4$ or $|y| > 5$

$\equiv$ either $|x| \geqslant 5$ or $|y| \geqslant 6$, as $x, y \in \mathbb{Z}$.

$$\equiv \text{ either } \begin{pmatrix} x \geqslant 5 \\ \text{or} \\ x \leq -5 \end{pmatrix} \text{ or } \begin{pmatrix} y \geqslant 6 \\ \text{or} \\ y \leq -6 \end{pmatrix}$$

$\equiv$ either $x \geqslant 5$ or $x \leq -5$ or $y \geqslant 6$ or $y \leq -6$

$\equiv$ either $-x+5 \leq 0$ or $x+5 \leq 0$ or $-y+6 \leq 0$ or $y+6 \leq 0$

$$\boxed{\begin{aligned} -x+5 &\leq M(1-t_1) \\ x+5 &\leq M(1-t_2) \\ -y+6 &\leq M(1-t_3) \\ y+6 &\leq M(1-t_4) \\ t_1+t_2+t_3+t_4 &\overset{\geqslant}{\leq} 1 \\ t_i \in \{0,1\}&, \quad i=1,2,3,4 \end{aligned}}$$

also works