MATH 364: Lecture 2 (08/22/2024)

Today: * Linear algebra review
- matrix transpose, rank, inverse

* Gauss-Jordan method in general

Example for Case 2(b) (for $A\bar{x}=\bar{b}$ with infinitely many solutions)

Consider the following system: $\left\{\begin{array}{l} x_1 + 2x_2 + 2x_3 = 6 \\ 3x_1 + 6x_2 + 5x_3 = 8 \end{array}\right\}$ $m=2,\ n=3$

Since there are $n=3$ variables, and only $m=2$ equations here, we will have at least one free variable.

$$\left[\begin{array}{ccc|c} \textcircled{1} & 2 & 2 & 6 \\ 3 & 6 & 5 & 8 \end{array}\right] \xrightarrow{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 6 \\ 0 & 0 & -1 & -10 \end{array}\right] \xrightarrow[\text{then } (-1)R_2]{R_1+2R_2} \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & -14 \\ 0 & 0 & \textcircled{1} & 10 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 &= -14 \\ x_3 &= 10 \end{aligned}$$

x_1, x_3 are basic

x_2 is free or non-basic

$$\underbrace{\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -14 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s,}_{\text{parametric vector form of all solutions}} \quad s \in \mathbb{R}$$

set of all real numbers

We can choose x_2 as any real value s, and for each choice, we get a (different) solution for the original system.

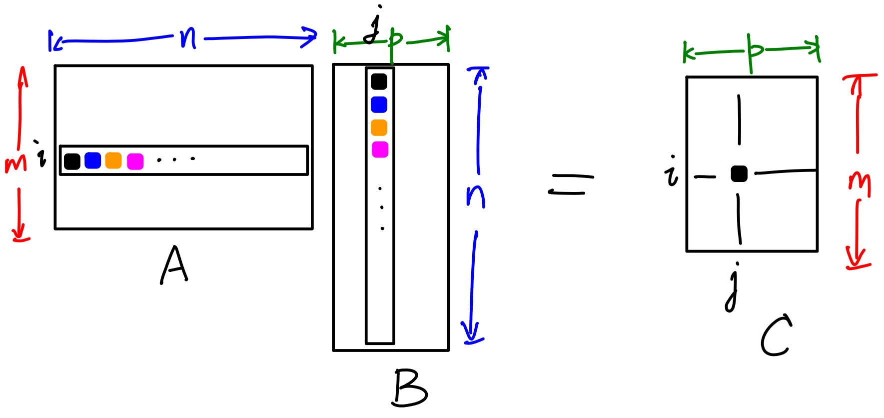
Transpose of a matrix $A \in \mathbb{R}^{m\times n}$

If $B = A^{T} \xrightarrow{} \text{transpose}$ then $B_{ij} = A_{ji}$ interchange rows and columns.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & -1 & 4 \end{bmatrix}_{2\times 3} \qquad A^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}_{3\times 2}$$

Matrix Multiplication

If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, then $C = AB$ is in $\mathbb{R}^{m\times p}$.



$C_{ij} = ?$

$$C_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + A_{i3}B_{3j} + \cdots + A_{in}B_{nj} = \sum_{k=1}^{n} A_{ik}B_{kj}.$$

Rules of matrix multiplication

* $AB \neq BA$ typically (BA might not even be defined)
  $\to$ not symmetric
* $(AB)C = A(BC)$ $\longrightarrow$ is associative
* $(AB)^T = B^T A^T$

⋮ (several more)

Linear Independence (LI) of vectors

Let $V = \{\bar{v}_1, \dots, \bar{v}_n\}$, where $\bar{v}_j \in \mathbb{R}^m$ → set of m-vectors with real entries

"Definition"

Def A **linear combination** of vectors in V is a vector

$$\bar{u} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n, \quad \text{where } c_j \in \mathbb{R} \ \forall j.$$

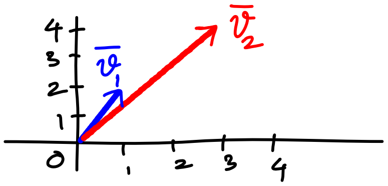
"for all"

If $c_j = 0$ for all j, $\bar{u}$ is the zero vector. This is the **trivial linear combination** of the vectors in V.

Def The vectors in V are **linearly independent** (LI) if the only linear combination of those vectors that is equal to the zero vector is the trivial linear combination.

e.g., $\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

$\bar{v}_1$ and $\bar{v}_2$ are not along the same line



If $c_1\bar{v}_1 + c_2\bar{v}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, what are c_1, c_2?

Solve for c_1, c_2 (as a system of linear equations):

$$\left[\begin{array}{cc|c} \textcircled{1} & 3 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{then } R_2(-\frac{1}{2})]{R_2-2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

The unique solution is $c_1 = c_2 = 0$. Hence $\{\bar{v}_1, \bar{v}_2\}$ is LI.

Def If there is a nontrivial linear combination of $\bar{v}_j$'s that is the zero vector, then V is **linearly dependent** (LD).

e.g., $\bar{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix}-3\\-6\end{bmatrix}$, then $3\bar{v}_1 + \bar{v}_3 = \begin{bmatrix}0\\0\end{bmatrix}$, showing $\{\bar{v}_1, \bar{v}_2\}$ is LD.

← zero vector

Note: If $\bar{0} \in V$, then V is LD.

Say $\bar{v}_1 = \bar{0}$. Then $c_1\bar{v}_1 + 0\bar{v}_2 + 0\bar{v}_3 + \cdots + 0\bar{v}_n = \bar{0}$ for any $c_1 \neq 0$ is a non-trivial linear combination that is the zero vector.

Rank of a matrix

Def The **rank** of $A \in \mathbb{R}^{m \times n}$ is the size of a largest LI subset of its rows or its columns.

Def rank(A) = # pivot columns in echelon form of A.

Examples

$A = \begin{bmatrix}1 & 2 & 0\\2 & 4 & 1\end{bmatrix}$; rank(A) = 2. e.g., $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is an LI subset of columns

$C = \begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ rank(C) = 1. $\left(\begin{bmatrix}2\\4\end{bmatrix} = 2\begin{bmatrix}1\\2\end{bmatrix}\right)$.

$O = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$ rank(O) = 0, as $\begin{bmatrix}0\\0\end{bmatrix}$ is LD by itself,

↓ big "Oh"

Because $c_1\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ for any $c_1 \neq 0$.

↙ Also, we noted above that any set that contains $\bar{0}$ is LD.

How to tell if $V=\{\bar{v}_1,\ldots,\bar{v}_n\}$, $\bar{v}_j\in\mathbb{R}^m$, is LI?

- If $n>m$, V is LD → more vectors than # entries in each of them $\Rightarrow$ LD.
- If $n\leq m$, then form $A=[\bar{v}_1\,\bar{v}_2\cdots\bar{v}_n]$ ($m\times n$ matrix) and find $\text{rank}(A)$ ($=$ # pivot columns in echelon form of A)
  - if $\text{rank}(A)<n$ then V is LD
  - if $\text{rank}(A)=n$ then V is LI.

e.g., $V=\{\bar{v}_1,\bar{v}_2,\bar{v}_3\}=\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}2\\0\\4\end{bmatrix},\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$. Is V LI?

$$A=\begin{bmatrix}\textcircled{1}&2&3\\0&0&1\\2&4&0\end{bmatrix}\xrightarrow{R_3-2R_1}\begin{bmatrix}1&2&3\\0&0&\textcircled{1}\\0&0&-6\end{bmatrix}\xrightarrow{R_3+6R_2}\begin{bmatrix}\textcircled{1}&2&3\\0&0&\textcircled{1}\\0&0&0\end{bmatrix}$$

$\text{rank}(A)=2$, So V is LD.

Notice that one need not go to the reduced row echelon form of A to identify the number of pivot columns – echelon form will do. In simpler words, $\text{rank}(A)=$ # pivot columns in A.

Inverse of a matrix

→ square matrix

Def For $A \in \mathbb{R}^{m\times m}$, if there is another matrix $B \in \mathbb{R}^{m\times m}$ such that

$AB = BA = I_m$, then B is the **Inverse** of A.

↪ m×m identity matrix

We denote this fact by $B = A^{-1}$. Similarly, $A = B^{-1}$.

Here, we say that A is **invertible**.

Why study inverses?

For $A\bar{x} = \bar{b}$ with $A \in \mathbb{R}^{m\times m}$ and invertible, we can do

$$A^{-1}(A\bar{x} = \bar{b})$$

multiply by A^{-1} on the left (on both sides)

"implies" $\Rightarrow (A^{-1}A)\bar{x} = A^{-1}\bar{b}$

$$\Rightarrow I\bar{x} = A^{-1}\bar{b} \quad \text{or} \quad \bar{x} = A^{-1}\bar{b}$$

Thus, knowing A^{-1} we can solve $A\bar{x} = \bar{b}$ directly.

How to invert $A \in \mathbb{R}^{m\times m}$?

Use GJ!

$$\text{If } [A \mid I_m] \xrightarrow{\text{EROs}} [I_m \mid B], \text{ then } B = A^{-1}.$$

But if we do not get I_m in place of A, then A is not invertible.

e.g., $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow[\text{then } R_1 \rightleftarrows R_2]{R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right] \xrightarrow[\text{then } R_1 - 3R_2]{R_2 \times (-1)} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$B$$

$B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ is A^{-1}. Check: $AB = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. ✓

Can invert 2x2 matrices directly using formula:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\underbrace{ad - bc}_{\text{determinant}} \neq 0$, then $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

here $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $2 \times 3 - 1 \times 5 = 1 \neq 0$, so $A^{-1} = \dfrac{1}{1} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

Here we are solving two systems $A\bar{x} = \bar{b}_1$ and $A\bar{x} = \bar{b}_2$ for the same A matrix simultaneously. More generally, for $A \in \mathbb{R}^{m \times m}$, we solve m systems of the form $A\bar{x} = \bar{e}_j$, $j = 1, \ldots, m$, where $\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{\text{th}}$ position is the j^{th} unit vector (or the j^{th} column of the identity matrix I_m).

Gauss-Jordan (GJ) Method in general

$A \in \mathbb{R}^{m\times n}$, $\bar{b} \in \mathbb{R}^m$.

$$[A|\bar{b}] \xrightarrow{\text{EROs}} \begin{array}{c} \\ r \\ m-r \end{array} \left[\begin{array}{cc|c} & \overset{n-r}{\longleftrightarrow} & \\ I_r & \tilde{N} & \tilde{\bar{b}}_1 \\ O & O & \tilde{\bar{b}}_2 \end{array}\right]$$

zero matrices

Here, $\operatorname{rank}(A) = r$.

1. If $\tilde{\bar{b}}_2 \neq \bar{0}$ (at least one entry is nonzero), then the system is inconsistent.
2. If $\tilde{\bar{b}}_2 = \bar{0}$, we can ignore the last $(m-r)$ rows of zeros.

   Assume the variables are split such that

   $\bar{x} = \begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix}$, where $\bar{x}_B$ are the r basic variables and $\bar{x}_N$ are the $n-r$ non-basic variables.

$\left[\begin{array}{cc|c} I_r & \tilde{N} & \tilde{\bar{b}}_1 \end{array}\right]$ gives

$$I_r \bar{x}_B + \tilde{N}\bar{x}_N = \tilde{\bar{b}}_1$$

$$\Rightarrow \quad \bar{x}_B = \tilde{\bar{b}}_1 - \tilde{N}\bar{x}_N$$

free vars!

If we set $\bar{x}_N = \bar{z}$ ($n-r$ vector of parameters), this is the parametric vector form!