MATH 364: Lecture 3 (08/27/2024)

Today:
* GJ method, example
* hints on HW1 problems
* LP formulations

The Relevant case of Gauss-Jordan Method

We consider the case when $\text{rank}(A) = m = \#$ rows of A, i.e., when none of the equations are redundant. We get

$$[A \,|\, \bar{b}] \xrightarrow{GJ} [I_m \;\tilde{N} \,|\, \tilde{\bar{b}}]$$

We do not get the zero matrices at the bottom. Also, $\tilde{\bar{b}}_1 = \tilde{\bar{b}}$.

We can split A into $[\underbrace{B}_{m} \; \underbrace{N}_{(n-m)}]$, where B are all the pivot columns.

$\text{rank}(B) = m$, so B is invertible. So B^{-1} exists.

$A\bar{x} = \bar{b}$ is equivalent to $[B \; N]\begin{bmatrix} \bar{x}_B \\ \bar{x}_N \end{bmatrix} = \bar{b}$, i.e.,

$$B^{-1}\left(B\bar{x}_B + N\bar{x}_N = \bar{b}\right)$$

$$\Rightarrow \quad \underbrace{B^{-1}B}_{I_m}\,\bar{x}_B + B^{-1}N\,\bar{x}_N = B^{-1}\bar{b}$$

$$\Rightarrow \quad \bar{x}_B = B^{-1}\bar{b} - B^{-1}N\bar{x}_N$$

$$= \tilde{\bar{b}}_1 - \tilde{N}\bar{x}_N.$$

Again, this is the parametric vector form of the solutions.

Example from Lecture 2:

$$\left\{\begin{array}{l} x_1 + 2x_2 + 2x_3 = 6 \\ 3x_1 + 6x_2 + 5x_3 = 8 \end{array}\right\} \qquad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -14 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s, \quad s \in \mathbb{R}$$

basic variables

Let us take $BV = \{x_1, x_3\}$, $NBV = \{x_2\}$ (as given to us).
Then we can split A as follows into $[B \ N]$.

$$A = \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \atop \begin{bmatrix} 1 & \boxed{2} & 2 \\ 3 & \boxed{6} & 5 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \ N = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \text{ and } \bar{x}_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}, \bar{x}_N = [x_2].$$

B

N

recall, for $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$B^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$B^{-1} = \frac{1}{(1\times 5 - 3\times 2)} \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

$$\Rightarrow \tilde{b} = B^{-1}\bar{b} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} -14 \\ 10 \end{bmatrix}, \quad \tilde{N} = B^{-1}N = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \bar{x}_B = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \tilde{b} - \tilde{N}s = \begin{bmatrix} -14 \\ 10 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} s, \quad s \in \mathbb{R}.$$

Combining with $x_2 = s$ (free variable), we can write

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -14 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} s, \quad s \in \mathbb{R},$$ which is what we got originally.

Problems from Homework 1

1. (a) Show $B = A + A^T$ is symmetric

B is symmetric if $B^T = B$.

$$B^T = (A+A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B.$$

follow from properties of matrix transpose

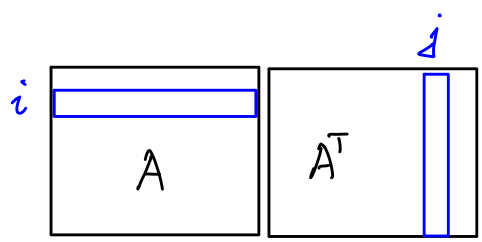
OR

Show $B_{ij} = B_{ji}$ (for B to be symmetric)

$$B_{ij} = A_{ij} + [A^T]_{ij} = A_{ij} + A_{ji} = A_{ji} + A_{ij} = B_{ji}.$$

(b) $B = AA^T$

$B_{ij} \stackrel{?}{=}$



$$B_{ij} = \sum_{k=1}^{n} A_{ik} \cdot A^T_{kj} = \sum_{k=1}^{n} A_{ik} A_{jk} = \sum_{k=1}^{n} A_{jk} A_{ik} = B_{ji}$$

OR

$$B^T = (AA^T)^T = (A^T)^T A^T = AA^T = B.$$

since $(AB)^T = B^T A^T$ in general.

You could try on small instances, e.g., 2x2, 2x3, etc, to identify the pattern or rule, but you must present general arguments as above.

4\. Recall: to find inverse of B, we apply GJ to

$$[B|I] \xrightarrow{\text{EROs}} [I|B^{-1}]$$

(b) $B \xrightarrow{2R_1} B'$.

$$[B'|I] \xrightarrow{\text{EROs}} [I|\,?\,]$$ → in terms of B^{-1} ?

Can start with $[B|I] \xrightarrow{2R_1} [B'|I']$ almost I

Try to argue how B^{-1} changes if we started with the additional ERO $(2R_1)$.

<u>Another approach: Elementary matrices</u>

$B \xrightarrow{2R_1} B'$ means $B' = EB$, where $E = \begin{bmatrix} 2 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$. → elementary matrix

$I \xrightarrow{2R_1} E$ (apply same ERO to identity)

Then use the result on inverse of product of matrices: $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B')^{-1} = (EB)^{-1} = B^{-1}E^{-1}$$

Explain what the effect of multiplying B^{-1} by E^{-1} will be.

$$E^{-1} = \begin{bmatrix} 1/2 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} \text{ here.}$$

Linear Optimization Formulations

We study how to create models for optimization problems arising in many different real life situations. The typical scenarios we work with involve minimizing costs or maximizing revenue or profit subject to meeting demands, or meeting limits on available resources.
These models are also called linear programming (LP) formulations.

The main defining criterion is for us to be able to write the objective function and constraints as **linear functions** or **(in)equalities** of the variables. We illustrate the process on an example.

(Taken from *Introduction to Mathematical Programming* by Winston and Venkataramanan.)

Farmer Jones must decide how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. Wheat can be sold at $4 per bushel, and corn at $3 per bushel. Seven acres of land and 40 hours of labor per week are available. Government regulations require that at least 30 bushels of corn need to be produced in each week. Formulate and solve an LP which maximizes the total revenue that Farmer Jones makes.

→ we will do this part later

There is no algorithm (or, step-by-step rules to follow) using which one could write every LP. We list some guidelines here. As you become more familiar with such problems, you will be able to do them more directly (rather than follow a step-by-step procedure).

0. Make notes of various numbers mentioned in the problem

This step is highly recommended, at least for the first several LP formulations you write.

	corn	wheat	availability
land	→	→	7 acres
labor hrs	4 hr/wk	10 hr/wk	40 hours
yield	10 bu/acre	25 bu/acre → bushels/acre	
selling price	\$3/bu	\$4/bu	
restriction	x		30 bu/wk

1. Define the **decision variables** (d.v.'s)

Let x_1 = # acres of corn
x_2 = # acres of wheat

} it is important to declare the d.v.'s explicitly (as done here). Saying "x_1 = corn", for instance, will not work!!

Goal: Express the objective and constraints (restrictions) as linear functions or (in)equalities of these d.v.s.

2. Define the **objective function**

Usually, maximize revenue/profit, minimize cost, etc.

Goal: maximize total revenue here

Notice that the units for each term is \$: (price/bu) × (bu/acre) × (# acres)

$$\text{maximize} \quad \underbrace{\$3}_{\text{price \$/bu}} \cdot \underbrace{10}_{\text{yield \#bu/acre}} \cdot \underbrace{x_1}_{\text{\# acres}} + \$4 \cdot 25 \cdot x_2 \qquad \text{(total revenue)}$$

In short, $\max z = 30x_1 + 100x_2$ (total revenue)

max: "maximize"

30, 100: objective function coefficients (of x_1, x_2)

convention: we usually denote the objective function as z (more explanation coming later on!)

You **must** include a short explanation in parentheses for the objective function, and for each (set of) constraint(s) you write. Later on, when we introduce the software AMPL, we could use these explanations to denote the constraints and the objective function.

3. Define constraints

Constraint 1: land availability

$$x_1 + x_2 \le 7 \quad \text{(land availability)}$$

acres of corn

acres of wheat

total land available

Constraint 2: labor hrs

$$4x_1 + 10x_2 \le 40 \quad \text{(labor hrs)}$$

hrs/acre of corn

acres of corn

Constraint 3: Government regulation

$$10x_1 \ge 30 \quad \text{(min. corn)}$$

"at least"

bushels/acre of corn

acres of corn

4. Define sign restrictions on variables

x_1, x_2 are # acres, so negative values do not make sense.

$$x_1, x_2 \ge 0 \quad \text{(non-negativity)}.$$

There are scenarios where negative values might make perfect sense. for example, when modeling a budget balance, a deficit could be a negative value, and a surplus is a positive value.

If no explicitly mentioned as non-negative, the variables are assumed to be unrestricted in sign (urs).

Putting it all together, we get the (entire) LP formulation:

$$
\begin{aligned}
\max \quad & z = 30x_1 + 100x_2 && \text{(total revenue)} \\
\text{s.t.} \quad & x_1 + x_2 \le 7 && \text{(land availability)} \\
& 4x_1 + 10x_2 \le 40 && \text{(labor hrs)} \\
& 10x_1 \ge 30 && \text{(min corn)} \\
& x_1, x_2 \ge 0 && \text{(non-negativity)}
\end{aligned}
$$

subject to