MATH 364: Lecture 4 (08/29/2024)

Today:
* alternative formulation for Farmer Jones LP
* assumptions of LP
* graphical solution in 2D

Alternative formulation for Farmer Jones LP

(Taken from *Introduction to Mathematical Programming* by Winston and Venkataramanan.)

Farmer Jones must decide how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. Wheat can be sold at \$4 per bushel, and corn at \$3 per bushel. Seven acres of land and 40 hours of labor per week are available. Government regulations require that at least 30 bushels of corn need to be produced in each week. Formulate and solve an LP which maximizes the total revenue that Farmer Jones makes.

$$\left.\begin{array}{llll} \max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\ & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\ & 10x_1 & \geq 30 & \text{(min corn)} \\ & x_1,\ x_2 & \geq 0 & \text{(non-negativity)} \end{array}\right\}$$

Formulation with x_1 = # acres of corn, x_2 = # acres of wheat

We now consider a different set of d.v.'s.

Let x_i = $\underbrace{\text{\# bushels of crop } i,\ i=1,2,\ 1=\text{corn},\ 2=\text{wheat}}$.

such short-hand notation will be handy in many cases!

$$\begin{array}{llll} \max & z = 3x_1 + 4x_2 & & \text{(total revenue)} \\ \text{s.t.} & \left(\frac{1}{10}\right)x_1 + \underbrace{\left(\frac{1}{25}\right)x_2}_{\text{\# acres of wheat}} & \leq 7 & \text{(land availability)} \\ & 4\left(\frac{1}{10}\right)x_1 + 10\left(\frac{1}{25}\right)x_2 & \leq 40 & \text{(labor hrs)} \\ & x_1 & \geq 30 & \text{(min corn)} \\ & x_1,\ x_2 & \geq 0 & \text{(non negativity)} \end{array}$$

acres/bu of corn

Naturally, the two formulations are equivalent — as you will confirm in homework 2.

Assumptions of LP Formulations

1. **Proportionality**: Contribution of a variable to objective function or to the left-hand side of a constraint is proportional to its value.

   This assumption is central to linearity. Hence we get terms of the form $30x_1$, $10x_2$, etc., and not $5x_1^2$ or $\frac{6}{x_j}$.

2. **Additivity**: Total contribution from all variables is the sum of the contributions from individual variables.

   Another assumption central to linearity. Hence we get expressions of the form $30x_1 + 100x_2$, $4x_1 + 10x_2$, etc., and not $30x_1/100x_2$, for instance.

3. **Divisibility**: The variables can take fractional values, e.g., can farm corn in 2.8 acres.

   But we might insist on x_j to take only integer values in some cases, e.g., whether to reopen the government, modeled by a binary variable, for which fractional values do not make sense.

   Then we get integer programming (IP), which we will introduce later.

4. **Certainty** All coefficients and right-hand side values are known beforehand with surety.

   In stochastic LP, we assume probabity distributions on some of the data (objective function or constraint coefficients, or rhs numbers).

   Another approach is robust optimization, where we want to find solutions that are optimal over ranges of values of the input parameters.

How do we solve LPs? We extend the ideas used to solve systems of linear equations in 2D to solve LPs in 2D, to start with. Then we talk about extending these ideas to higher dimensions.

Graphical method to solve LPs in 2D

We illustrate the procedure on the original formulation of the Farmer Jones LP.

$$\begin{array}{llll} \max & z = 30x_1 + 100x_2 & & \text{(total revenue)} \\ \text{s.t.} & x_1 + x_2 & \leq 7 & \text{(land availability)} \\ & 4x_1 + 10x_2 & \leq 40 & \text{(labor hrs)} \\ & 10x_1 & \geq 30 & \text{(min. corn)} \\ & x_1, x_2 & \geq 0 & \text{(non-negativity)} \end{array}$$

The first step is to plot the feasible region of the LP.

Def The set of all $(x_1, \dots, x_n)$ satisfying all constraints, including sign restrictions is the **feasible set** of the LP. It is also called the **feasible region**.

We start by plotting the feasible region of the Farmer Jones LP by plotting the constraints. → all constraints are inequalities here, including nonnegativity constraints.

How to plot $x_1 + x_2 \leq 7$?

First, plot $x_1 + x_2 = 7$ (can use $A(7,0)$ and $B(0,7)$ as two points)
↳ need two points to plot a straight line

Then we pick the "correct" side to plot the "$\leq$" constraint, by testing any one point. The obvious choice is to use $(0,0)$. Here, $0+0 \leq 7$, and hence $(0,0)$ is on the correct side. We indicate the correct side by drawing arrow(s).

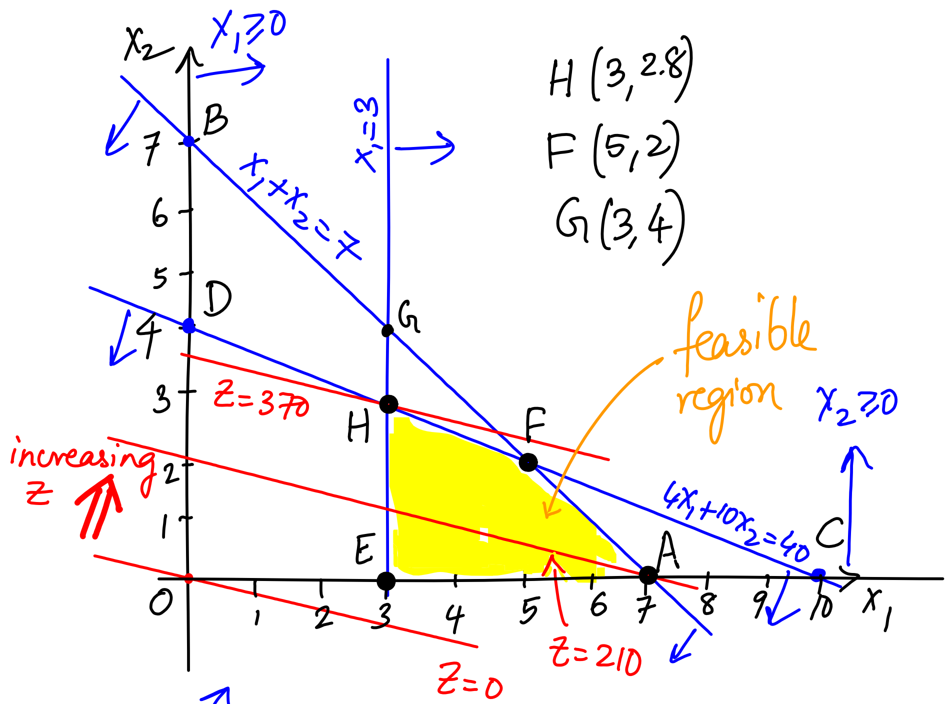
$4x_1 + 10x_2 \leq 40$

We plot the "=" line using $C(10,0)$ and $D(0,4)$. Again, $(0,0)$ is on the correct side for the "$\leq$" inequality.

$10x_1 \geq 30$

We need only one point, e.g., $E(3,0)$, as the line $x_1 = 3$ is vertical. Here, $(0,0)$ is on the wrong side of the "$\geq$" inequality.

$x_1, x_2 \geq 0$ } gives the first quadrant



(Such rough figures will do for graphical solutions of LPs!

The region AEHF is the feasible region of the LP, which is the intersection of all the half-spaces.

H: point of intersection of $x_1 = 3$ and $4x_1 + 10x_2 = 40$, i.e., $H(3, 2.8)$.

It is critical to solve for these points of intersection, rather than trying to guess them from the rough diagram.

F: $x_1 + x_2 = 7$
$4x_1 + 10x_2 = 40$
$F(5,2)$

H: $x_1 = 3$
$4x_1 + 10x_2 = 40$
$H(3, 2.8)$

Note that there are infinitely many points in the feasible region here.
or feasible points

Def A point in the feasible region is a **feasible solution**. Any point not in the feasible region is an **infeasible solution**.

An infeasible solution violates at least one constraint, e.g., $G(3,4)$ is an infeasible point, as it violates $4x_1 + 10x_2 \leq 40$.

Def An **optimal solution** is a solution (or point) in the feasible region at which the objective function is optimum, i.e., it is maximum for a max ~~ob~~ objective function, and minimum for a min objective function.

How to find an optimal solution? → We involve the objective function in the picture now.

We plot a line corresponding to one value of z. Notice that for any value of z, the objective function is a straight line, of the form $30x_1 + 100x_2 = z$.

For $z = 210$, $(7,0)$ and $(0,2.1)$ are two points.

picked 210, as it's a multiple of 30, and hence could find a point on it quickly.

Then we change the value of z, i.e., plot lines parallel to the first line. Since only the rhs value is changing, the slope remains the same, i.e., $\frac{-100}{30}$.

right-hand side

A parallel z-line through the origin has $z=0$. Hence the good direction here is to slide the z-line up (since we want to maximize z).

Looks like we could slide the z-line up to F, and then to H. Remember that we need to stay within the feasible region, and hence cannot go any further than H.

$F(5,2)$: $z = 30x_1 + 100x_2 = 30\times5 + 100\times2 = 350$

$H(3,2.8)$: $z = 30\times3 + 100\times2.8 = 370$. ✓

In some cases, it will be obvious which corner point is the optimal solution even from the rough diagram. But in other cases, we need to verify using calculations.

So, $H(3,2.8)$ is the optimal solution.

Interpretation: Jones needs to farm corn in 3 acres and wheat in 2.8 acres, to get a maximum total revenue of \$370/wk.

Note: 3 acres of corn are required to produce ≥ 30 bushels of corn.

He is using $3+2.8=5.8$ out of 7 acres, and

$4(3)+10(2.8)=40$ labor hours.

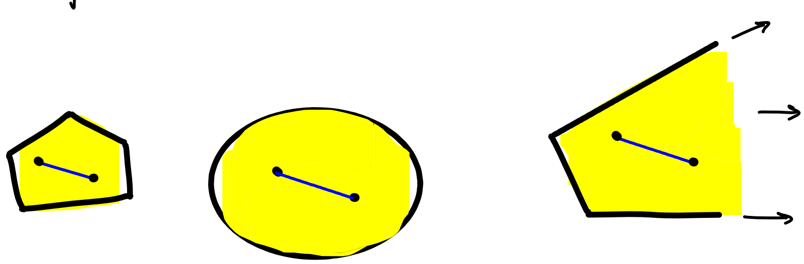
So, the (labor hrs) constraint is satisfied as an equation.

Def A constraint that is satisfied as an equation at the optimal solution is a **binding constraint**. A constraint that is satisfied as a strict inequality at the optimal solution is a **non-binding constraint**.

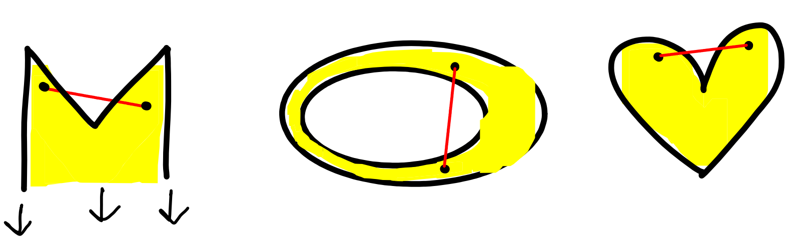
At $H(3, 2.8)$, the (labor hrs) and (min corn) constraints are binding, while the (land availability) constraint is non-binding.

It turns out that we need to look at only the vertices of the feasible region for the optimal solution. This result follows from the fact that the feasible region of an LP is a **convex set**.

Def A set S is **convex** if the line segment joining any two points in S lies entirely inside S.



convex



not convex