MATH 364: Lecture 5 (09/03/2024)

Today: * cases of LP
* one full example

Hw2, Problem 1

Candidate choices for d.v.'s:

1. x_i = # hrs in paint shop for toy i, $i=1,2$, $1 \equiv$ dirty, $2 \equiv$ ugly
2. x_i = # toy i (per day) i, $i=1,2$, $1 \equiv$ dirty, $2 \equiv$ ugly
3. x_{ij} = # hrs of toy i in shop j, $i,j=1,2$; $i=1 \equiv$ dirty, $i=2 \equiv$ ugly
$j=1 \equiv$ assembly,
$j=2 \equiv$ paint

Think proportionality!

1500 Dirty C's per day in assembly shop $\Rightarrow$
$\left(\frac{1}{1500}\right)$ day of assembly is required for each Dirty C.

For assembly shop, can use it for assembling Dirty C's or Ugly C's all day.

(total time for Dirty C's) + (total time for Ugly C's) ≤ 1 (assembly)

1 day

Similar constraint for paint shop.

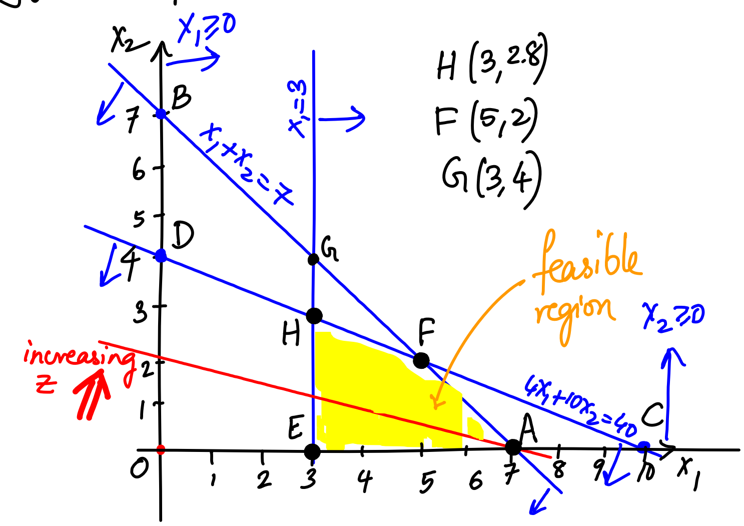
If x_i = # hrs in paint shop for toy i,

figure out the # toys of type i (using proportionality)

Objective function: $4(\text{\# Dirty C's}) + 3(\text{\# Ugly C's})$ (total profit)

Recall feasible region of Farmer Jones LP is AEHF.

AEHF is a convex region.



Def The corners of the feasible region of an LP are called as **extreme points** or **vertices**.

Because of its convexity and linearity (defined by linear inequalities), if an LP has an optimal solution and its feasible region has extreme points, an optimal solution will occur at an extreme point!

We now consider cases of LP, which correspond to the cases of systems of linear equations. Recall that such a system has a unique solution, infinitely many solutions, or no solutions

Cases of LP

Case 1 **Unique optimal solution**. If the LP has a unique optimal solution, that optimal solution will be a corner point of its feasible region.

e.g., Jones LP. H(3, 2.8) is the unique optimal solution.

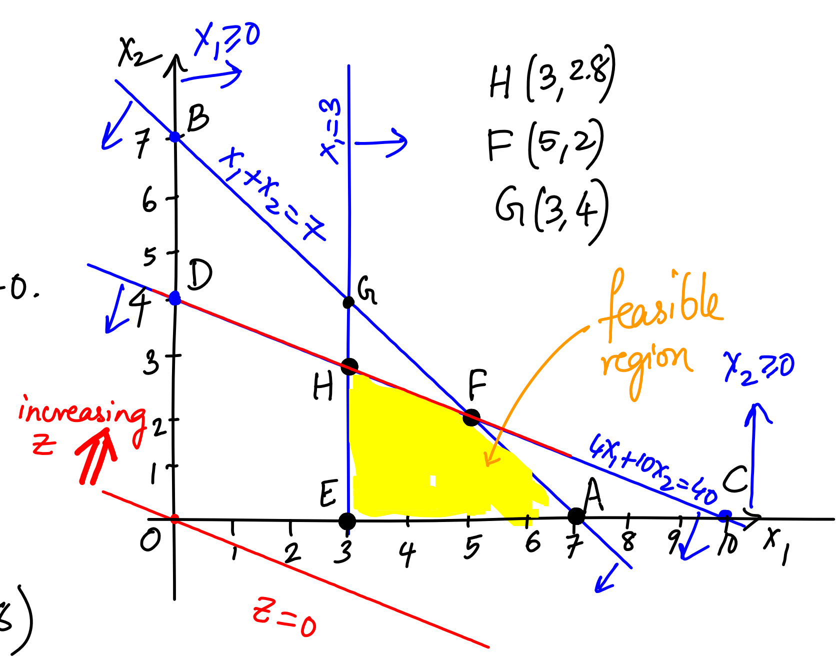
Case 1 is the good, typical case of LPs. But there are three other special cases of LP (Cases 2, 3, and 4).

Case 2 Some LPs have infinitely many optimal solutions, i.e., they have **alternative optimal solutions**.

For example, in the Jones LP, "if price of corn is also \$4/bushel", the revenue function becomes

$$\max \quad z = 40x_1 + 100x_2 \qquad \text{(total revenue)}$$

Note that slope of the z-line is now equal to the slope of the (labor hrs) line $4x_1+10x_2 \le 40$.



At F(5,2),

$$z = 40(5) + 100(2) = 400$$

We get $z=400$ at $H(3, 2.8)$ as well: $40(3) + 100(2.8) = 400$.

The same z-value is obtained at every point on $\overline{HF}$.

Note: The slope of z-line must be same as that of a binding constraint for the LP to be Case 2.

Case 3 Some LPs have no feasible solutions, and hence no optimal solutions. Such LPs are called as **infeasible LPs**.

Case 4 **Unbounded LPs**.

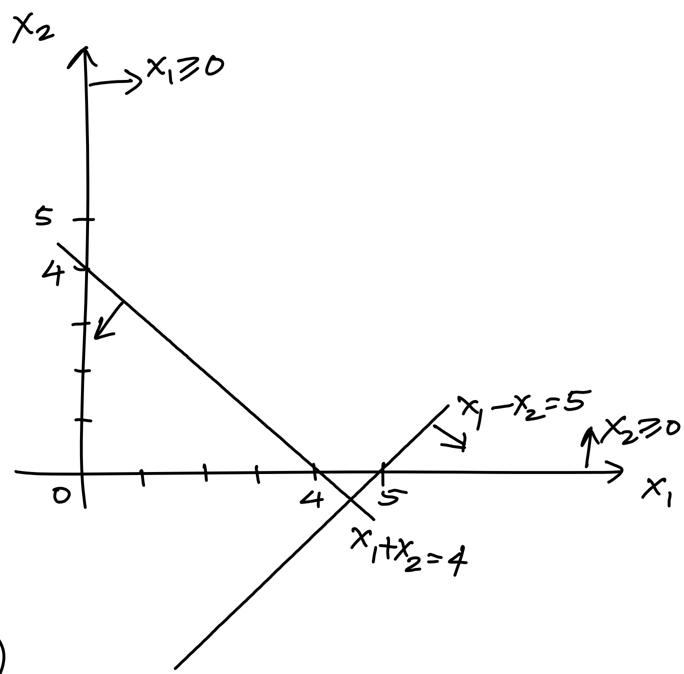
There are feasible solutions with arbitarily large z-values for max LPs or arbitrarily small z-values for min LPs. Hence an unbounded LP has no optimal solutions.

Cases 1, 2, and 3 correspond to the three cases of systems of linear equations ($A\bar{x}=\bar{b}$) — unique optimal solution, infinitely many solutions, and infeasible systems. Case 4 (unbounded LPs) is unique to LPs, i.e., there is no corresponding case in $A\bar{x}=\bar{b}$.

We now consider several example LPs, and identify which case each one belongs to.

LP instances

1. $$\begin{aligned} \max\ & z = x_1 + x_2 \\ \text{s.t.}\ & x_1 + x_2 \leq 4 \\ & x_1 - x_2 \geq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$



The feasible region is empty, i.e., it's a Case 3 LP (infeasible LP).

Typically, when we get an infeasible LP in practice, it indicates we do not have enough raw materials to satisfy all demands, for instance.

2. max $z = 4x_1 + x_2$

s.t. $8x_1 + 2x_2 \le 16$

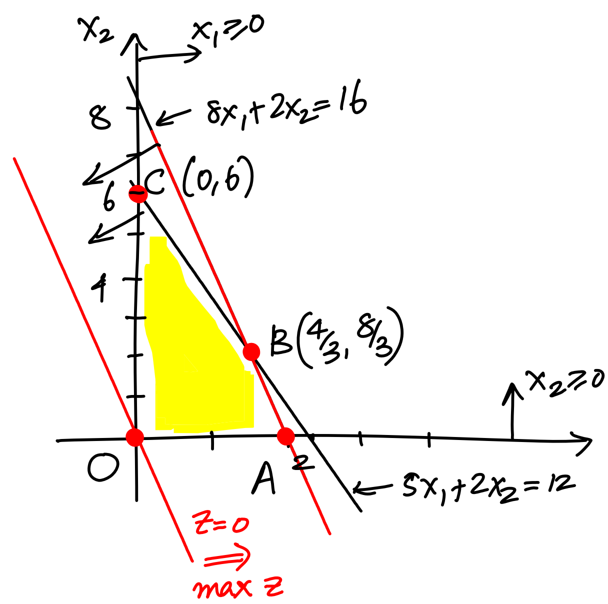
$5x_1 + 2x_2 \le 12$

$x_1, x_2 \ge 0$

B: $8x_1 + 2x_2 = 16$

$5x_1 + 2x_2 = 12$

$x_1 = \frac{4}{3},\ x_2 = \frac{8}{3}$



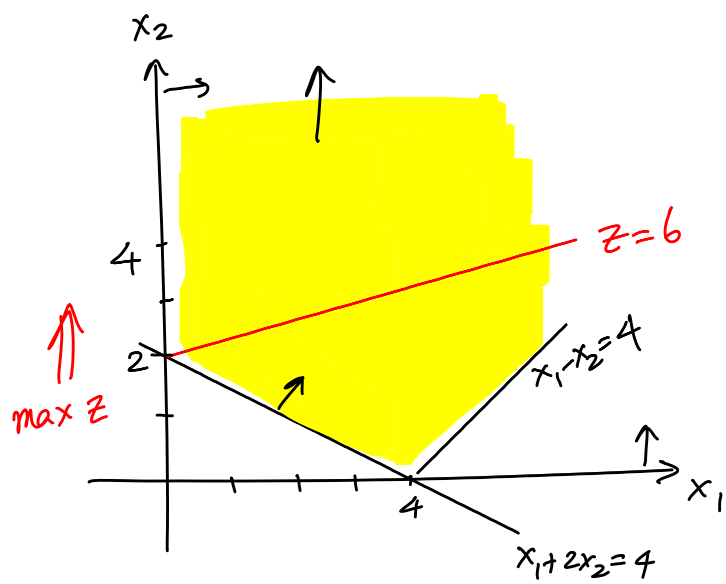
Every point on $\overline{AB}$ is an optimal solution. At each such point, $z = 4(2) + (0) = 8$. This is a Case 2 LP, i.e., it has alternative optimal solutions.

3. max $z = -x_1 + 3x_2$

s.t. $x_1 - x_2 \le 4$

$x_1 + 2x_2 \ge 4$

$x_1, x_2 \ge 0$



Can slide the z line up without any limit.

Case 4 LP.

A set (or region) is bounded if it can be enclosed in a finite box (can be large). It is unbounded if no such finite box exists.

One Full example

Formulate and solve LP:

Richy Rich trades currencies, and is working with the Crooner, the currency of ImaginationLand, and the US Dollar (USD). He can buy Crooners at the rate of \$0.20 USD per Crooner, and can buy USD at the rate of 3 Crooner per USD. Let x_1 be the number of USD bought by paying Crooners, and x_2 the number of Crooner bought by paying USD. Assume all transactions happen simultaneously, and the only restriction is that Richy should have nonnegative numbers of Crooners and USD at the end of all transactions. Formulate an LP that maximizes the total number of USD Richy has after all transactions. Graphically solve the LP, and comment on the solution.

d.V's $x_1 = \#$ USD, $x_2 = \#$ Crs (Crooners) at end

$$\max \quad z = x_1 - \tfrac{1}{5}x_2 \qquad \text{(\# USD remaining)}$$

$$x_1 - \tfrac{1}{5}x_2 \geq 0 \qquad \text{(nonneg \# USD)}$$

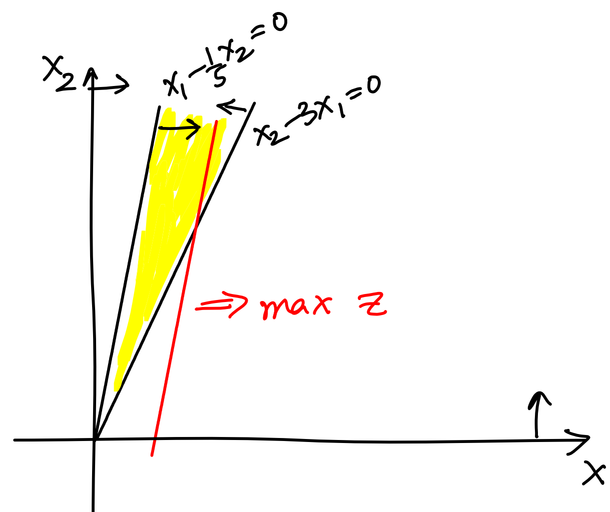
$x_2 \leq 5x_1$

$$x_2 - 3x_1 \geq 0 \qquad \text{(nonneg \# Crooners)}$$

$x_2 \geq 3x_1$

$$x_1, x_2 \geq 0$$

Feasible region extends upward without limit, and we could slide the z-line to the right without limit. Case 4 — unbounded LP.



The exchange rates given here are unreasonable, and will never be seen in real life.

$1 \text{ USD} \xrightarrow{\times 5} 5 \text{ Crs} \xrightarrow{\times \frac{1}{3}} \frac{5}{3} \text{ USD}$. $\rightsquigarrow$ So Richy could become infinitely rich!