MATH 364: Lecture 6 (09/05/2024)

Today: LP formulations

Linear Programming Formulation Problems

We introduce several LP formulation problems. They represent various scenarios from different application areas. The more such problems you are exposed to, the more comfortable you will get in tackling them.

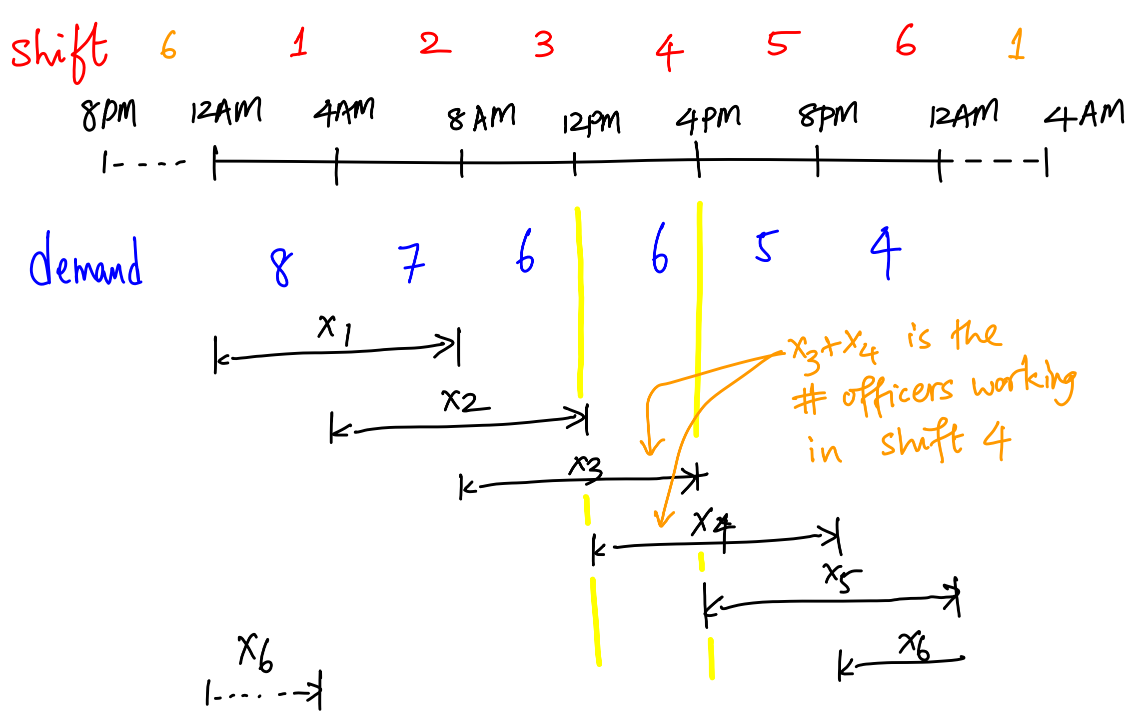
Staffing Problems

→ Winston-Venkataramanan: Intro to Math Programming

WV-IMP problem 2 Pg 75:

2 During each 4-hour period, the Smalltown police force requires the following number of on-duty police officers: 12 midnight to 4 A.M.—8; 4 to 8 A.M.—7; 8 A.M. to 12 noon—6; 12 noon to 4 P.M.—6; 4 to 8 P.M.—5; 8 P.M. to 12 midnight—4. Each police officer works two consecutive 4-hour shifts. Formulate an LP that can be used to minimize the number of police officers needed to meet Smalltown's daily requirements.

A sketch of the timeline (as shown below) is often helpful for such problems.



d.V.s

Let x_i = # officers starting duty in shift i, $i=1,\dots,6$
(x_6 = # officers working shifts 6 and 1 of next day)

$$
\begin{array}{llrl}
\min & z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 & & \text{(total \# officers)} \\
\text{s.t.} & x_1 \qquad\qquad\qquad\qquad + x_6 & \geq 8 & \text{(shift 1 req.)} \\
& x_1 + x_2 & \geq 7 & \text{(shift 2 req.)} \\
& \quad x_2 + x_3 & \geq 6 & \text{(shift 3 req.)} \\
& \qquad x_3 + x_4 & \geq 6 & \text{(shift 4 req.)} \\
& \qquad\quad x_4 + x_5 & \geq 5 & \text{(shift 5 req.)} \\
& \qquad\qquad x_5 + x_6 & \geq 4 & \text{(shift 6 req.)} \\
& \qquad\qquad\quad x_j & \geq 0,\ j=1,\dots,6 & \text{(non-neg)}
\end{array}
$$

subject to

It is not necessary to line up the columns of each x_j when you write such formulations. But this practice does help with the readability!

A point about the divisibility assumption for the Smalltown police LP. Ideally we need to insist that all x_j's are integers, as they model # cops. But in this case, the optimal solution will have integer values. This happens because a special property is satisfied. We will revisit this topic later.

Blending Problems

a class of problems where several raw materials are blended together to form products.

WV-IMP Problem 5, pg 92:

5 Chandler Oil Company has 5,000 barrels of oil 1 and 10,000 barrels of oil 2. The company sells two products: gasoline and heating oil. Both products are produced by combining oil 1 and oil 2. The quality level of each oil is as follows: oil 1—10; oil 2—5. Gasoline must have an average quality level of at least 8, and heating oil at least 6. Demand for each product must be created by advertising. Each dollar spent advertising gasoline creates 5 barrels of demand and each spent on heating oil creates 10 barrels of demand. Gasoline is sold for \$25 per barrel, heating oil for \$20. Formulate an LP to help Chandler maximize profit. Assume that no oil of either type can be purchased.

Assume conservation of mass/volume, e.g., 10 ba oil 1 + 15 ba oil 2 ≡ 25 ba gas

Assumption about blending: the volume of product is equal to the volumes of crudes (or raw materials) mixed, i.e., there is no volume lost.

	quality	ad	price		oil 1	oil 2
				qlty:	10	5
gas	≥ 8	5 ba/\$	\$25/ba			
heating oil	≥ 6	10 ba/\$	\$20/ba	qty:	5,000	10,000

decisions

1. how much gas and h. oil to make?
2. how much to spend on ads for gas & h. oil?

1a. # barrels of oil 1 & oil 2 used for making gas?
1b. # barrels of oil 1 & oil 2 used for making h. oil?

d.v.'s

x_{ij} = # barrels of oil i used to make product j, $i=1,2$, $j=g,h$ (gas, h. oil)

$x_{1g}, x_{1h}, x_{2g}, x_{2h}$

y_j = \$ spent on ads for product j, $j=g,h$ (y_g, y_h)

Objective function (maximize profit)

$$\max\ z = 25\underbrace{(x_{1g}+x_{2g})}_{\text{total \# ba. of gas}} + 20\underbrace{(x_{1h}+x_{2h})}_{\text{total vol. of h. oil}} - \underbrace{(y_g+y_h)}_{\text{cost for ads}} \quad \text{(net profit)}$$

We could use extra variables to model the total amount of gas and total amount of heating oil. But we still need the split variables x_{ig}, x_{ih}, $i=1,2$.

Constraints

Limit on availability of oils 1 and 2:

$$x_{1g} + x_{1h} \leq 5000 \quad \text{(oil 1 avail.)}$$

$$x_{2g} + x_{2h} \leq 10{,}000 \quad \text{(oil 2 avail.)}$$

Meet demand generated by ads:

$$x_{1g} + x_{2g} \geq \underbrace{5y_g}_{\text{barrels of demand for gas generated by spending \$}y_g\text{ on ads}} \quad \text{(demand for gas)}$$

$$x_{1h} + x_{2h} \geq 10y_h \quad \text{(demand for h. oil)}$$

Average quality:

$$\underbrace{\frac{10\cdot x_{1g} + 5\cdot x_{2g}}{x_{1g} + x_{2g}}} \geq 8 \quad \text{(quality of gas)}$$

average quality of gasoline – the average is taken over the volume mixed, as a weighted average

$$\frac{10 \cdot x_{1h} + 5 \cdot x_{2h}}{x_{1h} + x_{2h}} \geq 6 \quad \text{(quality of h.oil)}$$

You could leave these constraints as is — no need to simplify.

Note that the quality constraints are indeed linear: cross multiply!

$$\frac{10 \cdot x_{1g} + 5 \cdot x_{2g}}{x_{1g} + x_{2g}} \geq 8 \Rightarrow \quad 10x_{1g} + 5x_{2g} \geq 8(x_{1g} + x_{2g}) \Rightarrow$$

$$2x_{1g} - 3x_{2g} \geq 0.$$

But you need not necessarily do this step of simplification — as long as you are sure of the linearity. In fact, the software AMPL will do the simplification for you!

Sign restrictions: $x_{ij}, y_j \geq 0$ for all i, j (non-neg)

or all vars ≥ 0 (non-neg).

Here is the entire LP:

$$\max \ Z = 25(x_{1g} + x_{2g}) + 20(x_{1h} + x_{2h}) - (y_g + y_h) \quad \text{(net profit)}$$

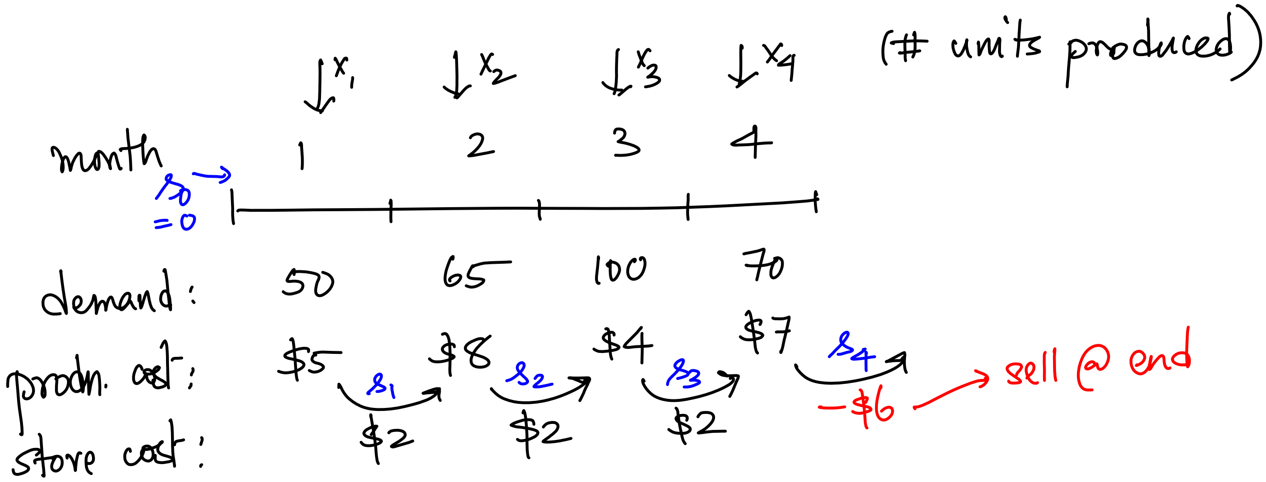
s.t.

$$\begin{aligned}
x_{1g} + x_{1h} &\leq 5000 && \text{(oil 1 avail.)} \\
x_{2g} + x_{2h} &\leq 10{,}000 && \text{(oil 2 avail.)} \\
x_{1g} + x_{2g} &\geq 5y_g && \text{(demand for gas)} \\
x_{1h} + x_{2h} &\geq 10y_h && \text{(demand for h.oil)} \\
(10 \cdot x_{1g} + 5 \cdot x_{2g})/(x_{1g} + x_{2g}) &\geq 8 && \text{(quality of gas)} \\
(10 \cdot x_{1h} + 5 \cdot x_{2h})/(x_{1h} + x_{2h}) &\geq 6 && \text{(quality of h.oil)} \\
\text{all vars} &\geq 0 && \text{(non-neg)}
\end{aligned}$$

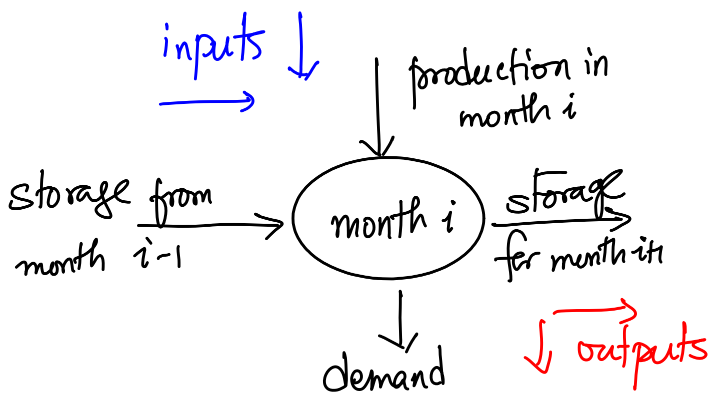
Inventory Planning

WV-IMP Pg 104, Problem 1:

1 A customer requires during the next four months, respectively, 50, 65, 100, and 70 units of a commodity (no backlogging is allowed). Production costs are \$5, \$8, \$4, and \$7 per unit during these months. The storage cost from one month to the next is \$2 per unit (assessed on ending inventory). It is estimated that each unit on hand at the end of month 4 could be sold for \$6. Formulate an LP that will minimize the net cost incurred in meeting the demands of the next four months.



For each month, we have "inflow = outflow" restriction, i.e., a flow-balance constraint.



We could produce some units in month i, and/or carry some over from month $(i-1)$. All these items are used to satisfy demand in month i, and whatever is left is carried over to month $i+1$.

(we'll finish the formulation in the next lecture...)