# MATH 364: Lecture 9 (09/17/2024)

Today: * LP in standard form
* basic solutions, bfs

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We will introduce the **simplex algorithm** to solve LPs with multiple variables using EROs. To apply this method, we first need to convert the LP into a standard $A\bar{x}=\bar{b}$ form — note that the input LP could have $\geqslant$, $\leqslant$, or $=$ constraints to start with.

**Def** An LP is in **standard form** if
1. all constraints are of the "$=$" form (equations); and
2. all variables are nonnegative ($\geqslant 0$).

So, no $\leqslant 0$ or unrestricted in sign variables in standard form

The objective function could be min or max.

Let's convert the following LP to standard form:

$$
\begin{aligned}
\min\ z = \ & 3x_1 + x_2 \\
\text{s.t.}\ & x_1 \geqslant 3 \\
& x_1 + x_2 \leqslant 4 \\
& 2x_1 - x_2 = 3 \\
& x_1, x_2 \geqslant 0
\end{aligned}
$$

Let's consider this constraint first.

We convert $x_1+x_2 \leqslant 4$ to an equation by adding a **slack variable** $s$ to the left-hand side, and adding non-negativity for $s$.

$$
\begin{aligned}
x_1 + x_2 + s &= 4 \\
s &\geqslant 0
\end{aligned}
$$

Note that $s \geqslant 0$ is required here. If $s=-1$, for instance, $x_1+x_2=5$, which violates the original constraint.

Recall: Farmer Jones LP:

$$x_1 + x_2 \le 7 \quad \text{(land available)}$$
$$4x_1 + 10x_2 \le 40 \quad \text{(labor hrs)}$$

$s_i \to$ slack variable for the $i^{th}$ constraint

$$x_1 + x_2 + \underbrace{s_1}_{\#\text{ acres unused}} = 7$$
$$4x_1 + 10x_2 + \underbrace{s_2}_{\#\text{ labor hrs unused}} = 40$$
$$s_1, s_2 \ge 0$$

Now, for $x_1 \ge 3$, we can write

$x_1 - e = 3$ and add $e \ge 0$.

Here $e$ is the **excess variable** (or surplus variable).

$$\begin{aligned} \min\ & z = 3x_1 + x_2 \\ \text{s.t.}\ & x_1 \ge 3 \\ & x_1 + x_2 \le 4 \\ & 2x_1 - x_2 = 3 \\ & x_1, x_2 \ge 0 \end{aligned} \quad \longrightarrow \quad \underbrace{\begin{aligned} \min\ & z = 3x_1 + x_2 \\ \text{s.t.}\ & x_1 - e = 3 \\ & x_1 + x_2 + s = 4 \\ & 2x_1 - x_2 = 3 \\ & x_1, x_2, s, e \ge 0 \end{aligned}}_{\text{LP in standard form}}$$

Slack/excess variables do not show up in the objective function.

↳ or, they have coefficient zero in the objective function.

## Another Example

$$\begin{aligned}
\max\ z = 20x_1 + 15x_2 & \\
\text{s.t.}\quad x_1 &\le 100 \quad s_1\\
x_2 &\le 200 \quad s_2\\
50x_1 + 35x_2 &\le 5000 \quad s_3\\
25x_1 + 15x_2 &\ge 2000 \quad e_4\\
x_1, x_2 &\ge 0
\end{aligned}$$

It is just convenient notation to use $s_i$ for slack variable of $i^{th}$ constraint, and $e_j$ for the excess variable of $j^{th}$ constraint.

↳ But one could instead use $x_3, x_4, x_5, x_6$ as the slack/excess variables here!

$$\begin{aligned}
\max\ z = 20x_1 + 15x_2 & \\
\text{s.t.}\quad x_1 + s_1 &= 100\\
x_2 + s_2 &= 200\\
50x_1 + 35x_2 + s_3 &= 5000\\
25x_1 + 15x_2 - e_4 &= 2000\\
x_1, x_2, s_1, s_2, s_3, e_4 &\ge 0
\end{aligned}$$

If input LP has $A\bar{x} \begin{pmatrix} \le \\ \ge \\ \vdots \end{pmatrix} \bar{b}$, then the standard form LP will be $[A\ I']\bar{x}' = \bar{b}$ where

$I'$ is "almost identity" matrix, and $\bar{x}' = \begin{bmatrix} \bar{x} \\ \bar{s} \end{bmatrix}$ → original vars → slack/excess vars.

If all constraints are $\le$, then we get $[A\ I]$, where $I$ is the $m \times m$ identity matrix (assuming there are $m$ constraints).

We will talk about the second condition (of requiring all variables to be $\geq 0$) later on. For now, assume all variables are $\geq 0$ to start with.

Once in the standard form, notice that the constraints all form a system $A\bar{x}=\bar{b}$. If the system has a unique solution, there is nothing more to do — that solution is the optimal solution. If $A\bar{x}=\bar{b}$ is inconsistent, then the LP is infeasible. The interesting case happens when $A\bar{x}=\bar{b}$ has free variables, and then we will involve the objective function to choose a best solution from among the infinitely many solutions possible.

An LP in standard form is
$$\begin{aligned} \max\ & \bar{c}^T\bar{x} \\ \text{s.t.}\ & A\bar{x}=\bar{b} \\ & \bar{x}\geq\bar{0} \end{aligned}$$

<u>Recall</u>: GJ for solving $A\bar{x}=\bar{b}$, $A\in\mathbb{R}^{m\times n}$, $\operatorname{rank}(A)=m\leq n$.

$$[A\,|\,\bar{b}] \longrightarrow [B\ N\,|\,\bar{b}] \xrightarrow{\text{EROs}} [I_m\ \tilde{N}\,|\,\tilde{\bar{b}}]$$

With $\bar{x}_B$ as the $m$ basic variables, and $\bar{x}_N$ as the $(n-m)$ non-basic (or free) variables, we get with $\bar{x}=\begin{bmatrix}\bar{x}_B\\ \bar{x}_N\end{bmatrix}$

$$\bar{x}_B + \tilde{N}\bar{x}_N = \tilde{\bar{b}}$$
$$\Rightarrow\ \bar{x}_B = \tilde{\bar{b}} - \tilde{N}\bar{x}_N$$

Choosing $\bar{x}_N=\bar{\alpha}$ (vector of parameters), we get $\bar{x}_B=\tilde{\bar{b}}-\tilde{N}\bar{\alpha}$.

The solution $\bar{x}$ obtained by setting $\bar{x}_N=\bar{\alpha}=\bar{0}$ is called a **basic solution** of $A\bar{x}=\bar{b}$.

$\bar{x}_N=\bar{0} \Rightarrow \bar{x}_B=\tilde{\bar{b}}$, so $\bar{x}=\begin{bmatrix}\tilde{\bar{b}}\\ \bar{0}\end{bmatrix}$ is a basic solution.

# How to find (a) basic solution(s)?

1. Choose $m$ basic variables (BV), which correspond to $m$ LI columns of $A$. when $n > m$, there could be many subsets of $m$ vars that are basic.
2. Set the remaining $(n-m)$ non-basic vars (NBV) to 0.
3. Solve for the $m$ basic variables.

## Example

$$\begin{aligned} x_1 + x_2 \quad &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad m=2, \; n=3$$

1. $BV = \{x_1, x_2\}$, $NBV = \{x_3\}$. Set $x_3 = 0$, solve for $x_1, x_2$:

$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 &= -1 \end{aligned} \qquad x_1 = 2, x_2 = 1$$

So, Basic solution is $\bar{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

2. $BV = \{x_1, x_3\}$, $NBV = \{x_2\}$.

Basic solution is $\bar{x} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ if the input system were that of an LP, this solution violates feasibility, as $x_3 \ngeq 0$.

3. $BV = \{x_2, x_3\}$, $NBV = \{x_1\}$. $x_1 = 0$

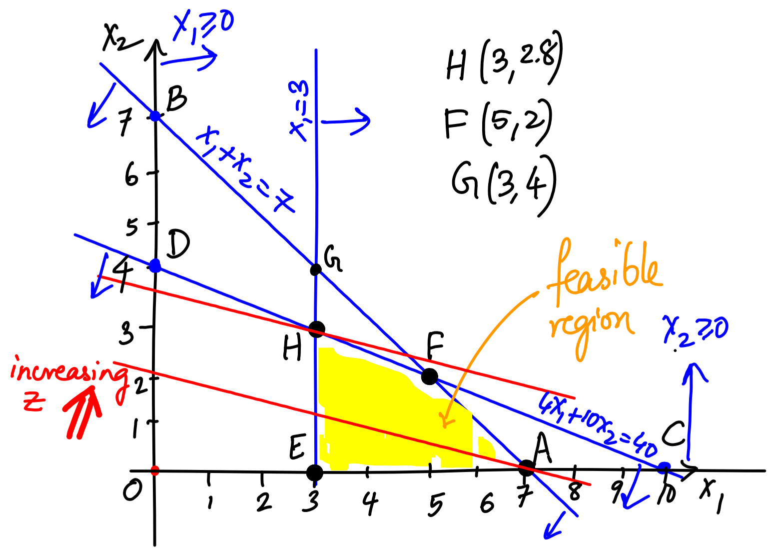
Basic solution is $\bar{x} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$.

**Def** For an LP in standard form, a basic solution in which all variables are nonnegative is a **basic feasible solution** (bfs).

Why study bfs's?

Recall: feasible region of an LP is a convex set.

**Result** If an LP in standard form has an optimal solution, then a corner point is guaranteed to be optimal.



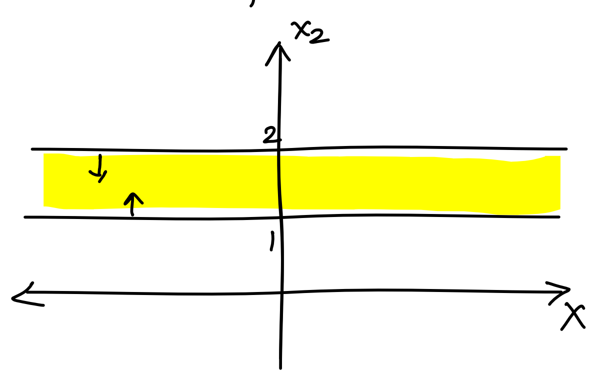

Q. Does every feasible LP have a corner point? No!

Consider max $x_2$

s.t. $1 \leq x_2 \leq 2$

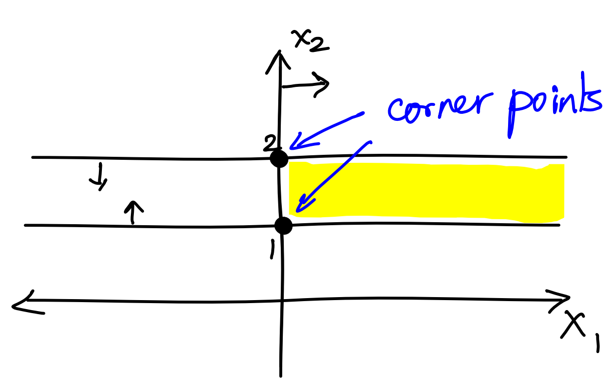
$x_1$ urs

unrestricted in sign
could be $\geq 0$ or $\leq 0$




There are no corner points here. The LP is indeed not unbounded. Any $(x_1, x_2)$ with $x_2 = 2$ is an optimal solution (Case 2).

If we add $x_1 \geq 0$, we get an LP in standard form, and we get two corner points!



**Result** Every LP in standard form has corner point(s).

**Result** A point in the feasible region of an LP in standard form is a corner point <u>if and only if</u> it corresponds to a bfs.

So, **corner point $\equiv$ bfs.**

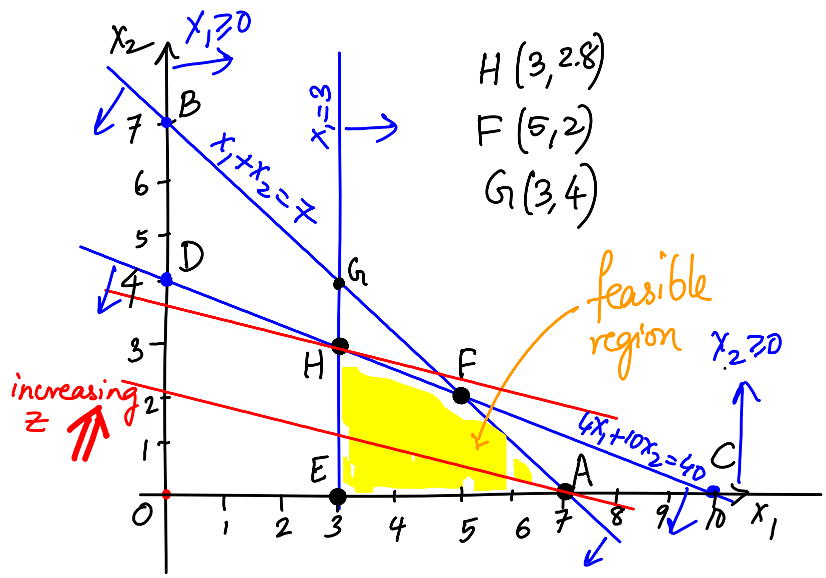
We demonstrate this correspondence for the Farmer Jones LP:

## Farmer Jones LP

$$\max\ z = 30x_1 + 100x_2$$

s.t.

$$x_1 + x_2 \le 7 \quad s_1$$
$$4x_1 + 10x_2 \le 40 \quad s_2$$
$$10x_1 \ge 30 \quad e_3$$
$$x_1, x_2 \ge 0$$




## Standard form

$$\max\ z = 30x_1 + 100x_2$$

s.t.

$$x_1 + x_2 + s_1 = 7$$
$$4x_1 + 10x_2 + s_2 = 40$$
$$10x_1 - e_3 = 30$$
$$x_1, x_2, s_1, s_2, e_3 \ge 0$$

$m = 3,\ n = 5$

$\text{rank} = m = 3$

$$\begin{aligned} x_1 + x_2 + s_1 \quad &= 7 \\ 4x_1 + 10x_2 \quad + s_2 \quad &= 40 \\ 10x_1 \quad - e_3 &= 30 \end{aligned}$$

1. $BV = \{x_1, x_2, s_1\}$, $NBV = \{s_2, e_3\}$

Setting $s_2 = e_3 = 0$, and solving we get $x_1 = 3$, $x_2 = 2.8$, $s_1 = 1.2$.

$H(3, 2.8)$

$\bar{x} = \begin{bmatrix} 3 \\ 2.8 \\ 1.2 \\ 0 \\ 0 \end{bmatrix}$ is a bfs, and corresponds to the vertex $H(3, 2.8)$