Introduction to Analysis I (Fall 2025)
Final Examination

- There are **six** problems in this exam, all presented in the next page.
- The total points (given in parentheses) add to 100.
- This is a **LIMITED OPEN RESOURCES** and **CLOSED COMMUNICATION and INTERNET** exam. You **are allowed** to use the course textbook (LSIRA), all documents posted on the course web page, as well as your own notes and your homework submissions for the class. But you are **not permitted** to use the internet or any AI engines or to communicate with any one about the exam. In particular, you are not allowed to use AI-assisted search engines or LLMs such as ChatGPT, Gemini, Claude, CoPilot, etc. for help/clarification on problems from the exam.
- You **must start your exam** by writing down the following statement word-by-word, and signing under the same.

  > I promise that I will not seek help from any person or any internet resource including any AI- or LLM-based resource while working on this exam. I will use only the allowed resources of textbook, course web page, and my own notes while working on the same.
  > —*Signature*

- You **must end your exam** by writing down the following **second** statement word-by-word, and again signing under the same.

  > As promised, I did not use any help from another person or online or from any AI-/LLM-based resource while working on this exam.
  > —*Signature*

- You **must email your submission as a SINGLE PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name should identify you** in the usual manner. If you are Tuong Lu Kim, you should name your submission TuongKim_Final.pdf (and **NOT** Tuong_Kim*.pdf or TuongLu Kim*.pdf or ...). You could add anything more to your filename *after* these terms, e.g., TuongKim_Final_Math401.pdf. Please avoid white spaces in the file name :-).
- **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, e.g., "TuongKim Final Exam submission".**
- This exam must be emailed to me **by 11:59 PM on Tuesday, December 9, 2025**.

1. (18) For a set X and two subsets $A, B \subseteq X$, define the function $d : X \times X \to \mathbb{R}$ as follows, where Δ denotes the symmetric difference of two sets.

$$d(A, B) = |A \Delta B| = |(A \setminus B) \cup (B \setminus A)| \,.$$

Prove that d defines a metric on X.

2. (16) Recall that two metrics d and d' are *equivalent* if the collection of open sets $\mathcal{O}$ in the metric space (X, d) is equal to the collection of open sets $\mathcal{O}'$ in the metric space (X, d'). Show that the metrics d and d' are equivalent if and only if the following conditions hold.

   (i) For every open set $O \in \mathcal{O}$ and for each $x \in O$, there exists an open d'-ball such that $B_{d'}(x; r') \subset O$ for $r' > 0$; and

   (ii) for every open set $O' \in \mathcal{O}'$ and for each $x' \in O'$, there exists an open d-ball such that $B_d(x'; r) \subset O'$ for $r > 0$.

3. Let (X, d) be a metric space and $Y \subset X$. Prove the following statements.

   (a) (10) If X is separable, then Y is separable.

   (b) (6) If Y is separable and is dense, then X is separable.

4. (17) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that f is uniformly continuous over $[a, b] \subset \mathbb{R}$. (Hint: Use the open cover property, given as the Heine-Borel theorem, over $\mathbb{R}$.)

5. (17) Let $f, g \in C([a, b], \mathbb{R})$, i.e., they are continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Define the metric d as follows:

$$d(f, g) = \int_a^b |f(t) - g(t)| \, d(t) \,.$$

Is $(C([0, 1], \mathbb{R}), d)$ complete? If yes, give a proof (you can use standard theorems/results seen in class or in the book as part of your proof as long as you cite them clearly). If no, show that it is so clearly using example(s).

6. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \,.$$

   (a) (7) Show that the series converges uniformly on $[a, \infty)$ for all $a > 0$.

   (b) (9) Show that the series does not converge uniformly on $(0, \infty)$.