Introduction to Analysis I (Fall 2025)
Practice Final Examination

- There are **six** problems in this exam, all presented in the next page.
- The total points (given in parentheses) add to 100.
- This is a **LIMITED OPEN RESOURCES** and **CLOSED COMMUNICATION and INTERNET** exam. You **are allowed** to use the course textbook (LSIRA), all documents posted on the course web page, and your own notes for the class. But you are **not permitted** to use the internet or any AI engines or to communicate with any one about the exam. In particular, you are not allowed to use AI-assisted search engines or LLMs such as ChatGPT, Gemini, Claude, CoPilot, etc. for help/clarification on problems from the exam.
- You **must start your exam** by writing down the following statement word-by-word, and signing under the same.

  > I promise that I will not seek help from any person or any internet resource including any AI- or LLM-based resource while working on this exam. I will use only the allowed resources of textbook, course web page, and my own notes while working on the same.
  > —*Signature*

- You **must end your exam** by writing down the following **second** statement word-by-word, and again signing under the same.

  > As promised, I did not use any help from another person or online or from any AI-/LLM-based resource while working on this exam.
  > —*Signature*

- You **must email your submission as a SINGLE PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name should identify you** in the usual manner. If you are Ned Gerblanski, you should name your submission NedGerblanski_Final.pdf (and **NOT** Ned_Gerblanski*.pdf or EdGerblanski*.pdf or ...). You could add anything more to your filename *after* these terms, e.g., NedGerblanski_Final_Math401.pdf. Please avoid white spaces in the file name :-).
- **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, e.g., "NedGerblanski Final Exam submission".**
- This exam must be emailed to me **by 11:59 PM on Tuesday, December 9, 2025**.

1. Let (X, d) be a metric space. We define the following function:

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} \; \forall x, y \in X.$$

   (a) (13) Show that d' defines a metric on X.

   (b) (7) Show that d and d' are equivalent metrics by showing they have the same open sets.

2. (14) Show that the set $S = \{m + n\sqrt{2} \,|\, m, n \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.
   Hint: Argue first that the nontrivial work comes down to $[0, 1) \subset \mathbb{R}$. Consider the fractional part $x - \lfloor x \rfloor$ of $x \in \mathbb{R}$, and look at $S \cap [0, 1)$...

3. State whether each of the following statements is True or False. You must justify your response properly. If False, give a counterexample. If True, give a proof.

   (a) (8) Let A and B be compact subsets of a metric space (X, d). Then $A \cap B$ is compact.

   (b) (6) A uniformly continuous function $f : X \to Y$ maps bounded sets to bounded sets.

4. Let (X, d) be a compact metric space, and let $f : X \to X$ be a continuous function.

   (a) (13) Show that the function $g(x) = d(x, f(x))$ is continuous and has a minimum point.

   (b) (7) If we have $d(f(x), f(y)) < d(x, y) \; \forall \, x, y \in X, x \neq y$ in addition to originally stated conditions, show that f has a unique fixed point.

5. (18) Let (X, d) be a metric space and assume the sequence of continuous functions $\{f_n\}$ converges uniformly to f. Show that if $\{x_n\}$ is a sequence in X that converges to $x \in X$, then $\{f_n(x_n)\}$ converges to $f(x)$. Give an example to show this result may not hold if $\{f_n\}$ converges only pointwise to f.

6. (14) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on all intervals $[a, \infty)$ for $a > 1$.