Introduction to Analysis I (Fall 2025)
Practice Midterm Examination

- There are **six** problems in this exam, all presented in the next page.
- The total points (given in parentheses) add to 100.
- This is a **CLOSED RESOURCES** exam. You are not supposed to use any external resources—checking textbooks, notes, cheat/summary sheets, AI/LLM tools, phone and internet resources, or communicating with other people about the exam are all **not permitted**.
- You **must start your exam** by writing down the following statement word-by-word, and signing under the same.

  > I promise that I will not use any external resources while working on this exam. I will not search the internet for any hints on the problems in the exam, and I will not look at any textbook, notes, handouts, or use online search and online resources, including any AI/LLM tools. I will also not communicate with any one else about this exam while working on the same.
  > —*Signature*

- You **must end your exam** by writing down the following **second** statement word-by-word, and **again signing** under the same.

  > As promised, I did not use any external resources while working on this exam.
  > —*Signature*

- You **must email your submission as a SINGLE PDF file** to kbala@wsu.edu. You are welcome to write answers by hand, and scan the writings.
- Your **file name should identify you** in the usual manner. If you are Uncle Tricky, you should name your submission UncleTricky_Midterm.pdf (and **NOT** Uncle_Tricky or "Uncle Tricky" or ...). You could add anything more to your filename *after* these terms, e.g., UncleTricky_Midterm_Math401.pdf. Please avoid white spaces in the file name :-).
- **Begin the SUBJECT of your email submission with the same `FirstnameLastname`, e.g., "UncleTricky Midterm submission".**
- This exam must be emailed to me **before 10:00 PM on the day of the exam**.

1. (15) Let $\mathcal{A}$ be a family of sets, and let B be another set. Prove the following statements.

$$B \setminus \left(\bigcup_{A_i \in \mathcal{A}} A_i \right) = \bigcap_{A_i \in \mathcal{A}} (B \setminus A_i) \tag{1}$$

$$\left(\bigcap_{A_i \in \mathcal{A}} A_i \right)^c = \bigcup_{A_i \in \mathcal{A}} A_i^c \tag{2}$$

2. (15) For a given relation R on a set X, we define its *converse relation* as follows.

$$R^c = \{(y, x) \in X \times X \mid (x, y) \in R\}. \tag{3}$$

Let F be the relation defined by a function $f : \mathbb{R} \to \mathbb{R}$, i.e.,

$$F = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = f(x)\}. \tag{4}$$

Show that F^c defines (is) a *function* when f is injective (i.e., one-to-one). Recall that a relation R defines (is) a function if for each $(x, y) \in R$ the y related/assigned to x is *unique*.

3. (16) Let S be the collection of all infinite sequences whose terms are 0 or 1. Is the set S countable? You need to clearly justify your Yes/No response.

4. (17) Given vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^m$, show that

$$\|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n\| \;\geq\; \|\mathbf{x}_1\| - \|\mathbf{x}_2\| - \;\ldots\; - \|\mathbf{x}_n\|. \tag{5}$$

You can use the standard triangle inequality (for two vectors).

5. (17) Consider the sequence $\{a_n\}$ whose terms are defined as follows.

$$a_1 = 1,\; a_2 = 3, \;\text{ and }\; a_n = \frac{1}{2}\left(a_{n-1} + a_{n-2}\right) \;\text{ for } n \geq 3. \tag{6}$$

Show that this sequence is Cauchy using the definition of a sequence being Cauchy.

Hints: Recall that, "informally", a sequence is Cauchy if we can go "far out enough" into the sequence and get that the difference between *any* two terms that come after is arbitrarily small. You could try to identify what $|a_n - a_{n-1}|$ is first. Then try to use the triangle inequality to show the desired result for the difference of an arbitrary pair of terms.

6. (20) Use the definition of continuity (using $\epsilon, \delta > 0$) of a function *directly* to show that if $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous functions at $x = a$ and if $g(a) \neq 0$ then f/g is also continuous at $x = a$. You **cannot** use results we saw in class or homework about continuity of $1/g$ or of fg as part of your proof.