MATH401: Lecture 1 (08/19/2025)

This is Introduction to Analysis I

I'm Bala Krishnamoorthy (Call me Bala).

Today: * Syllabus, logistics → see the course web page for details
* proof techniques
  - contrapositive proof
  - proof by contradiction
  - proof by induction

Book: Lindstrøm: Spaces–An Intro to Real Analysis (LSIRA)

LSIRA 1.1

Logical statements and notation.

If A then B (or $A \Rightarrow B$) "implies"

$A \Rightarrow B$ typically does not mean $B \Rightarrow A$.

e.g., A: p a natural number, is divisible by 6

B: p is divisible by 3.

$A \Rightarrow B$ holds, but $B \not\Rightarrow A$ (B does not imply A),

e.g., $p=9$.

But if $A \Rightarrow B$ and $B \Rightarrow A$ hold, we say A $\underbrace{\text{if and only if}}_{\text{iff}}$ B, or

$A \Leftrightarrow B$ (or A is equivalent to B).

To prove $A \Leftrightarrow B$, we often prove $A \Rightarrow B$ and $B \Rightarrow A$

$(A \Leftarrow B)$ separately.

We start by reviewing certain standard techniques to construct proofs of mathematical statements.

1. Contrapositive Proof

To show $A \Rightarrow B$, equivalently show not $B \Rightarrow$ not A $(\neg B \Rightarrow \neg A)$.

"negation" or "not"

"If A happened then B happened"
This statement is equivalent to
"If B did not happen then A did not happen."

LSIRA 1.1 Prob 3. Prove the following Lemma.

Lemma 1 If n is a natural number such that n^2 is divisible by 3, then n is divisible by 3.

This is $A \Rightarrow B$ where A: $3 \mid n^2$ (n^2 is divisible by 3). (or 3 divides n^2)
B: $3 \mid n$ (n is divisible by 3).

Let's try to prove $A \Rightarrow B$ directly: $n^2 = 3k$ (as $n^2 \mid 3$) $\Rightarrow$ $n = \sqrt{3}\sqrt{k}$ (taking square root on both sides)

Hard to conclude that $n \mid 3$ ☹! Would have to argue $k \mid 3$, which is not obvious!

Let's try proving $\neg B \Rightarrow \neg A$.

$\neg B$: n is not divisible by 3.

$\Rightarrow$ $n = 3p+1$ or $n = 3q+2$, for $p, q \in \mathbb{N}$. (set of natural numbers)

Case 1. $n = 3p+1$

$$\Rightarrow n^2 = (3p+1)^2$$
$$= 9p^2 + 6p + 1$$
$$= 3(3p^2+2p)+1$$
$$= 3k+1 \text{ for } k = 3p^2+2p$$

$\Rightarrow n^2$ is not divisible by 3

Case 2. $n = 3q+2$

$$\Rightarrow n^2 = (3q+2)^2$$
$$= 9q^2 + 12q + 4$$
$$= 9q^2 + 12q + 3 + 1$$
$$= 3(\underbrace{3q^2+4q+1}_{=k'})+1$$
$$= 3k'+1$$

$\Rightarrow n^2$ is not divisible by 3.

Hence we have proved that if n is not divisible by 3, then n^2 is not divisible by 3. Hence, by the contrapositive, we have $n^2 \mid 3 \Rightarrow n \mid 3$. $\square$

Should we always try to build a contrapositive proof?
Not necessarily! In cases where $A \Rightarrow B$ could be concluded directly, the contrapositive argument might make life harder!
It is one of the different proof approaches that you should be aware of.

2. Proof by Contradiction

Assume opposite of what you want to prove, and end up with a contradiction (or an obviously wrong statement). Hence the original assumption must be wrong, i.e., you have proved the statement.

LSIRA 1.1 Prob 3 (continued) Prove the following Theorem.

Theorem 2 $\sqrt{3}$ is irrational.

Assume $\sqrt{3}$ is $\underbrace{\text{rational}}$. → the opposite of what you want to prove

$\Rightarrow \left(\sqrt{3} = \frac{p}{q}\right)^2$, $p, q \in \mathbb{N}$ with no common factors. → by definition, any positive rational number can be written in the form p/q as specified.

Let's square both sides, and cross multiply.

$\Rightarrow 3q^2 = p^2 \Rightarrow 3 \mid p^2$ (p^2 is divisible by 3).

Hence by Lemma 1, $3 \mid p$. Let $p = 3k$. ($k \in \mathbb{N}$). Plug $p = 3k$ back in:

$\Rightarrow 3q^2 = (3k)^2 = 9k^2$ (divide both sides by 3)

$\Rightarrow q^2 = 3k^2$, i.e., $3 \mid q^2$ (q^2 is divisible by 3).

Again by Lemma 1, $3 \mid q$.

Since we started with the assumption that p and q have no common factors

Thus p and q have a common factor of 3, which is a contradiction.

Hence $\sqrt{3}$ is irrational. $\square$

3. Proof by Induction

To show a statement $P(n)$ holds for all $n \in \mathbb{N}$,

1. show $P(1)$ holds;
2. Assume $P(k)$ holds for some $k \in \mathbb{N}$.
3. Show $P(k+1)$ holds under Assumption 2.

"for all"

Example

Show that $P(n) = 3 + 5 + \cdots + 2n+1 = n(n+2) \ \forall n \in \mathbb{N}$.

1. $P(1) = 3 = 1(1+2)$ (so $P(1)$ is true).
2. Assume $P(k) = k(k+2)$ for some $k \in \mathbb{N}$.
3. $P(k+1) = P(k) + 2(k+1) + 1 = P(k) + 2k+3$

$$
\begin{aligned}
&= k(k+2) + 2k+3 \quad \text{by induction assumption.} \\
&= k(k+2) + k + k+3 \\
&= k(k+3) + k+3 \\
&= (k+1)(k+3) = n(n+2) \text{ for } n = k+1.
\end{aligned}
$$

$\Rightarrow P(n) = n(n+2) \ \forall n \in \mathbb{N}$. $\square$

MATH 401: Lecture 2 (08/21/2025)

Today: * sets and operations

Sets and Operations (LS IRA 1.2)

Set: Collection of mathematical objects.

They can be finite, e.g., $\{2,5,9,1,6\}$, or infinite, e.g., $[0,1]$, the collection of all $x \in \mathbb{R}$ with $0 \leq x \leq 1$.

"element of" set of all real numbers

Given sets A, B we have

$A \subseteq B$: A is a subset of, or equal to, B.

$A \subset B$: A is a strict subset of B, i.e., there is at least one $x \in B$ such that $x \notin A$.
But $\forall x \in A$, $x \in B$ holds.

To prove $A = B$, we often prove $A \subseteq B$ and $A \supseteq B$ (or $B \subseteq A$).

Here are ~~some~~ standard sets we will use regularly.

ϕ: empty set.

$\mathbb{N} = \{1, 2, 3, \ldots\}$, set of all natural numbers

$\mathbb{R}$ = set of all real numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, set of all integers

$\mathbb{Q}$ = set of rational numbers, $\mathbb{C}$ = set of complex numbers.

$\mathbb{R}^n$: set of all real n-tuples, or n-vectors

Notation for sets: $[-2, 1] = \{x \in \mathbb{R} \mid -2 \leq x \leq 1\}$.

closed interval from -2 to 1

"such that"
could also use ":" instead of "|".

More generally, $A = \{a \in B \mid P(a)\}$.

bigger set than A

property

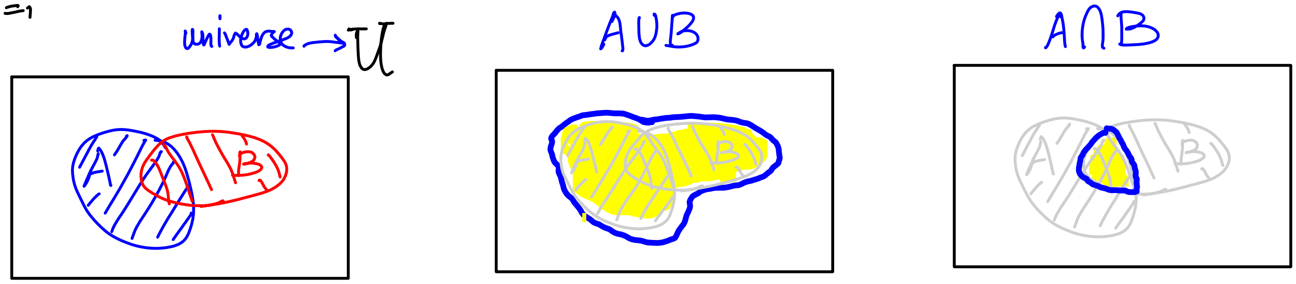
Union and Intersection

If A_i are sets for $i=1,\dots,n$, i.e., $A_1, A_2, \dots, A_n$ are sets, then

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{a \mid a \in A_i \text{ for at least one } i\}$ is their union,

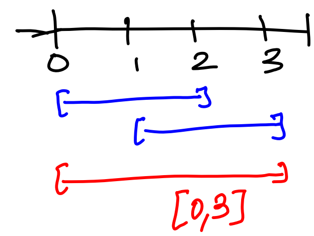
$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{a \mid a \in A_i\ \forall i\}$ is their intersection.

"for all"



LSIRA 1.2 Prob1

Show $[0,2] \cup [1,3] = [0,3]$.



We show $[0,2] \cup [1,3] \subseteq [0,3]$ and
$[0,2] \cup [1,3] \supseteq [0,3]$.

($\subseteq$) Let $x \in [0,2] \cup [1,3]$

$\Rightarrow x \in [0,2]$ or $x \in [1,3]$ (definition of $\cup$).

$x \in [0,2] \Rightarrow x \in [0,3]$ (as $[0,3]$ contains $[0,2]$)

$x \in [1,3] \Rightarrow x \in [0,3]$. In either case, $x \in [0,3]$.

Hence $[0,2] \cup [1,3] \subseteq [0,3]$.

($\supseteq$) Let $x \in [0,3]$. Hence $0 \le x \le 3$. Then we get that either $x \le 2$, and hence $x \in [0,2]$, or $x \in (2,3]$.

But if $x \in (2,3]$ then $x \in [1,3]$ (as $[1,3]$ includes $(2,3]$).

$\Rightarrow x \in [0,2] \cup [1,3]$.

Hence $[0,3] \subseteq [0,2] \cup [1,3]$. $\square$

The result is an obvious one. But we go through the steps of a formal proof more for practice!

Distributive Laws of Union and Intersection

For all sets $B, A_1, \ldots, A_n$, we have

LSIRA (1.2.1) $$B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) = (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n).$$

Using more compact notation, we can write

$$B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (B \cap A_i)$$

Proof

We will prove

$$B \cap (A_1 \cup \cdots \cup A_n) \subseteq (B \cap A_1) \cup \cdots \cup (B \cap A_n), \text{ and}$$

$$B \cap (A_1 \cup \cdots \cup A_n) \supseteq (B \cap A_1) \cup \cdots \cup (B \cap A_n).$$

('$\subseteq$') Let $x \in B \cap (A_1 \cup \cdots \cup A_n)$.

$\Rightarrow$ $x \in B$ and $x \in (A_1 \cup \cdots \cup A_n)$ (definition of $\cap$)

$\Rightarrow$ $x \in B$ and $x \in A_i$ for at least one A_i. (defn. of $\cup$)

$\Rightarrow$ $x \in B \cap A_i$ for at least one A_i.

$\Rightarrow$ $x \in (B \cap A_1) \cup \cdots \cup (B \cap A_n)$.

('$\supseteq$') Let $x \in (B \cap A_1) \cup \cdots \cup (B \cap A_n)$.

$\Rightarrow$ $x \in (B \cap A_i)$ for at least one A_i.

$\Rightarrow$ $x \in B$ and $\underbrace{x \in A_i \text{ for at least one } A_i}$

$\Rightarrow$ $x \in B$ and $x \in (A_1 \cup \cdots \cup A_n)$

$\Rightarrow$ $x \in B \cap (A_1 \cup \cdots \cup A_n)$. $\square$

LSIRA (1.2.2) is assigned in Homework 1.

Set Difference and Complement

We write $A \setminus B$ or $A - B$ "setminus"

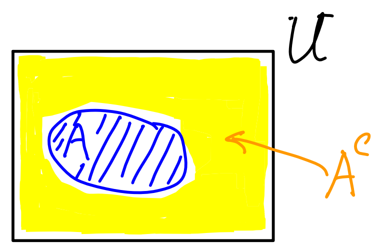
Caution! $*$ $A\setminus B \neq B \setminus A$!

"A setminus B" is $A \setminus B = \{a \mid a \in A, a \notin B\}$.

If $\mathcal{U}$ is the universe, i.e., $A \subseteq \mathcal{U}$ for all sets A, then

$$A^c = \mathcal{U} \setminus A = \{a \in \mathcal{U} \mid a \notin A\}$$ is the

complement of A (or A-complement).



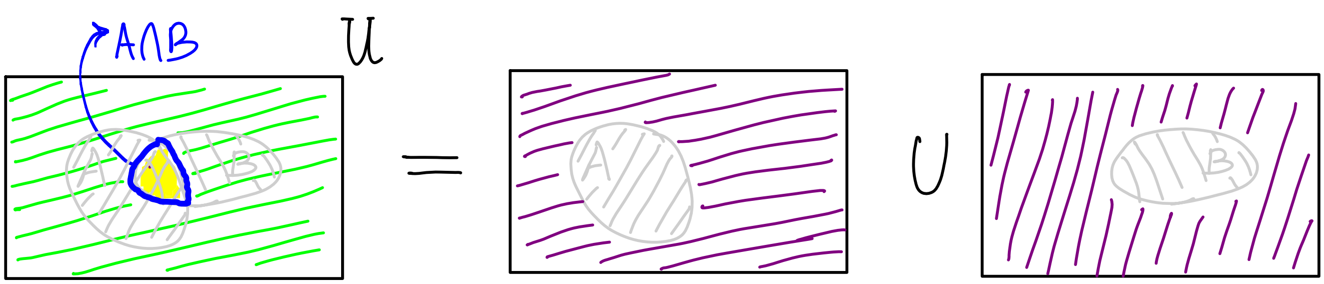
De Morgan's Laws

LSIRA (1.2.3) $(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$. "complement of union = intersection of complements"

LSIRA (1.2.4) $(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$. complement of intersection = union of complements.

See LSIRA for the proof.

Let's illustrate (1.2.4) for $n=2$, i.e., with A_1 and A_2 first.



$(A\cap B)^c$ $\qquad$ A^c $\qquad$ B^c

We will prove subset inclusion in both directions.

($\subseteq$) Let $x \in (A_1 \cap \cdots \cap A_n)^c$

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$ (definition of complement)

$\Rightarrow x \notin A_j$ for some j. (definition of $\cap$)

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \in A_1^c \cup \cdots \cup A_n^c$.

Hence $(A_1 \cap \cdots \cap A_n)^c \subseteq A_1^c \cup \cdots \cup A_n^c$.

($\supseteq$) Let $x \in A_1^c \cup \cdots \cup A_n^c$.

$\Rightarrow x \in A_j^c$ for some j

$\Rightarrow x \notin A_j$ for some j

$\Rightarrow x \notin A_1 \cap \cdots \cap A_n$. since $x \notin A_j$ for some j, it cannot be in the intersection of all A_i's.

$\Rightarrow x \in (A_1 \cap \cdots \cap A_n)^c$.

Hence $A_1^c \cup \cdots \cup A_n^c \subseteq (A_1 \cap \cdots \cap A_n)^c$. $\square$

Cartesian Products

For A, B : sets, we define

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$ → cartesian product of A and B

given A_i, $i = 1, \dots, n$ $(A_1, \dots, A_n)$, we define

→ compact notation $\prod$: product

$$A_1 \times A_2 \times \cdots \times A_n = \prod_{i=1}^{n} A_i = \{(a_1, \dots, a_n) \mid \underbrace{a_i \in A_i \ \forall i}_{a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n}\}.$$

e.g., $\mathbb{R}^n$: set of n-tuples of real numbers (or set of real n-vectors)

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

We'll finish the proof in the next lecture...

MATH401: Lecture 3 (08/26/2025)

Today: * families of sets, properties
* functions, images, pre images

We first do a problem on Cartesian products...

LSIRA 1.2 Prob 9 (Pg 11) Prove that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

'$\subseteq$' Let $(x,y) \in (A \cup B) \times C$.

$\Rightarrow x \in A \cup B,\ y \in C$ (Definition of cartesian product)

$\Rightarrow x \in A$ or $x \in B,\ y \in C$

If $x \in A$ then $(x,y) \in A \times C$, and

if $x \in B$ then $(x,y) \in B \times C$.

$\Rightarrow (x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow (x,y) \in (A \times C) \cup (B \times C)$.

'$\supseteq$' Let $(x,y) \in (A \times C) \cup (B \times C)$

$\Rightarrow (x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow x \in A,\ y \in C$ or $x \in B,\ y \in C \Rightarrow (x \in A$ or $x \in B),\ y \in C$.

$\Rightarrow x \in A \cup B,\ y \in C \Rightarrow (x,y) \in (A \cup B) \times C$.

$\square$

LSIRA 1.3 Families of Sets

Recall: $B \cap \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} (B \cap A_i)$. compact notation for distributive law (from Lecture 2)

We could write, instead, $B \cap \left(\bigcup_{i \in \mathcal{I}} A_i\right) = \bigcup_{i \in \mathcal{I}} (B \cap A_i)$, where $\mathcal{I} = \{1, 2, \dots, n\}$.

We now generalize $\mathcal{I}$ to be infinite in some cases, or indexing more general collections in general.

Def A collection of sets is a **family**.

e.g., $\mathcal{A} = \{[a,b] \mid a, b \in \mathbb{R}\}$ is the family of all closed intervals on $\mathbb{R}$.

Union and Intersection

We extend union, intersection, as well as their distribution to families.

$\bigcup_{A \in \mathcal{A}} A = \{a \mid a \in A \text{ for some } A \in \mathcal{A}\}$. collection of elements that belong to at least one set in the family

$\bigcap_{A \in \mathcal{A}} A = \{a \mid a \in A \text{ for all } A \in \mathcal{A}\}$ → collection of elements that belong to every set in the family.

We naturally extend distributive and De Morgan's laws to families . .

$$B \cap \left(\bigcup_{A \in \mathcal{A}} A\right) = \bigcup_{A \in \mathcal{A}} (B \cap A), \quad \left(\bigcap_{A \in \mathcal{A}} A\right)^c = \bigcup_{A \in \mathcal{A}} A^c, \text{ etc.}$$

We now work on some problems involving families of sets.

LSIRA 1.3 Prob 1 (Pg 12)

Show that $\bigcup_{n \in \mathbb{N}} [-n,n] = \mathbb{R}$.

('$\subseteq$') $\mathbb{R}$ is the universe here, so $\bigcup_{n \in \mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

Or, observe that $[-n,n] \in \mathbb{R}$ for each $n \in \mathbb{N}$, hence $\bigcup_{n\in\mathbb{N}} [-n,n] \subseteq \mathbb{R}$.

('$\supseteq$') Let $x \in \mathbb{R}$. To be more careful, we could consider $x=0$ separately. Note that $x=0 \in [-n,n]$ $\forall n \in \mathbb{N}$.

Let $m = \lceil |x| \rceil$, ceiling of absolute value of x, i.e., the smallest natural number $\geq |x|$.

$\lceil x \rceil = \text{ceil}(x)$ = smallest integer $\geq x$.

Then $x \in [-m,m] = [-\lceil |x| \rceil, \lceil |x| \rceil]$, as

$x \leq |x| \leq \lceil |x| \rceil = m$, and $x \geq -|x| \geq -\lceil |x| \rceil$.

e.g., $x = -3.3 \Rightarrow x \geq -|-3.3| = -3.3 \geq -4$.

$\Rightarrow x \in \bigcup_{n \in \mathbb{N}} [-n,n]$.

□

LSIRA 1.3 Prob 4

Show $\bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}] = \emptyset$ (empty set).

('$\supseteq$') $\emptyset \subseteq A$ for any set A (trivially).

('$\subseteq$') We show $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}] \subseteq \emptyset$.

$\emptyset^c = \mathbb{R}$. Hence we show $x \in \mathbb{R}$ is not in $\bigcap_{n\in\mathbb{N}} (0,\frac{1}{n}]$.

For $x \in \mathbb{R}$, we show $x \notin \bigcap_{n \in \mathbb{N}} (0, \frac{1}{n}]$.

If $x \leq 0$, then clearly, $x \notin (0, \frac{1}{n}]$ $\forall n \in \mathbb{N}$.

If $x \geq 1$, then $x \notin (0, \frac{1}{2}]$ for $n=2$, for instance.

Let $0<x<1$. Consider $m=\lceil \frac{1}{x}\rceil+1$.

Then $x \notin (0,\frac{1}{m}]$ as $x > \frac{1}{m} = \frac{1}{\lceil \frac{1}{x}\rceil+1}$. $\left(\lceil \frac{1}{x}\rceil+1 > \frac{1}{x}\right)$

$$\Rightarrow \quad x \notin \bigcap_{n\in\mathbb{N}} (0,\tfrac{1}{n}].$$

Q. Why take $\lceil \frac{1}{x}\rceil+1$?

Consider $x=\frac{1}{5}$, for instance.
Then $\lceil \frac{1}{x}\rceil = \lceil 5\rceil = 5$.
Hence $x \in (0,\frac{1}{m}]$ here!

$\square$

<u>LSIRA 1.3 Prob 5 (Pg 12)</u>

Prove that $B\cup\left(\bigcap_{A\in\mathcal{A}} A\right) = \bigcap_{A\in\mathcal{A}} (B\cup A)$.

($\subseteq$) Let $x\in B\cup\left(\bigcap_{A\in\mathcal{A}} A\right)$

$\Rightarrow$ $x\in B$ or $x\in \bigcap_{A\in\mathcal{A}} A$

$\Rightarrow$ $x\in B$ or $x\in A$ for each $A\in\mathcal{A}$.

$\Rightarrow$ $x\in B\cup A$ for each $A\in\mathcal{A}$.

$\Rightarrow$ $x\in \bigcap_{A\in\mathcal{A}}(B\cup A)$.

('$\supseteq$') Let $x\in \bigcap_{A\in\mathcal{A}}(B\cup A)$

$\Rightarrow$ $x\in B\cup A$ for every $A\in\mathcal{A}$.

$\Rightarrow$ $x\in B$ or $\underbrace{x\in A \text{ for every } A\in\mathcal{A}}$.

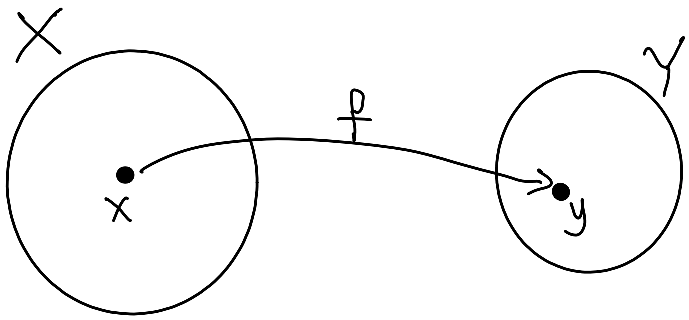
$\Rightarrow$ $x\in B$ or $x\in \bigcap_{A\in\mathcal{A}} A$ $\Rightarrow$ $x\in B\cup\left(\bigcap_{A\in\mathcal{A}} A\right)$.

$\square$

LSIRA 1.4 Functions

A function $f: X \to Y$ for sets X, Y is a rule that assigns for each $x \in X$ a **unique** $y \in Y$. We write $f(x) = y$, or

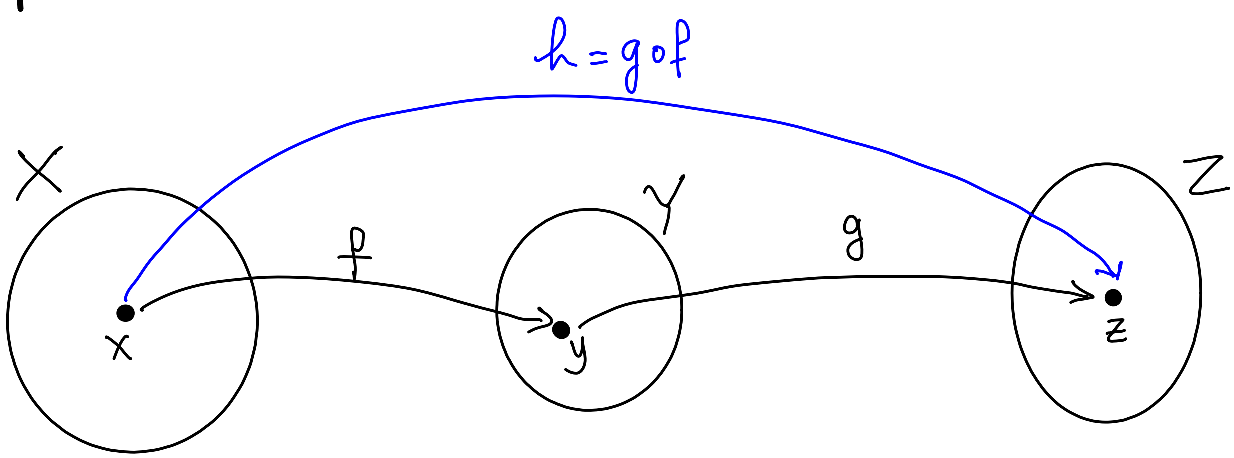
$x \mapsto y$ "maps to".



Rather than the graphs of functions you may have seen previously, we think of such visualizations for functions now.

X is the domain and Y the codomain of f.

Compositions

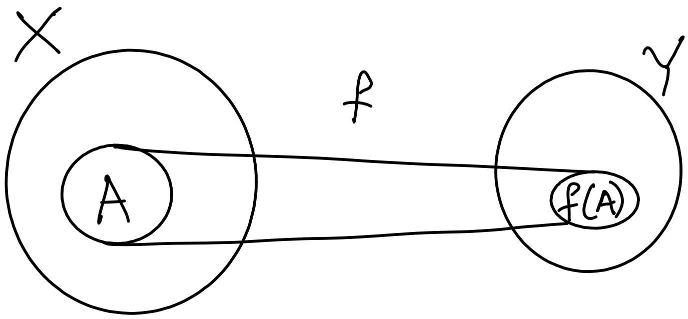


Let $f: X \to Y$ and $g: Y \to Z$ be functions. Then their composition is specified as $h: X \to Z$ defined as $h(x) = g(f(x))$.
The composition is written as $g \circ f$, with $g \circ f(x) = g(f(x))$.

"composition of f and g"

$f_1(f_2(\cdots f_n(x)))\cdots)$ → composition of n functions $f_1, f_2, \ldots, f_n$

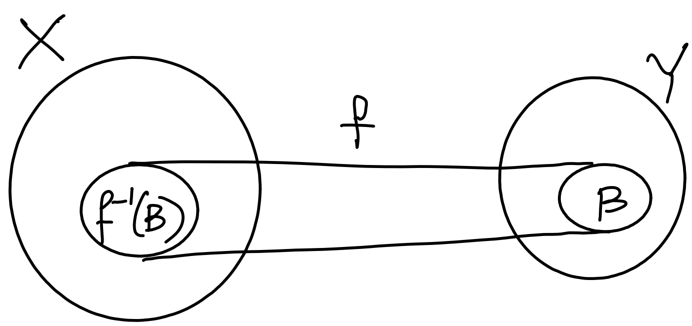
Function: $f: X \to Y$. We now define images and preimages under f.



For $A \subseteq X$, $f(A) \subseteq Y$ is defined as

$$f(A) = \{ f(a) \mid a \in A \},$$

and is called the **image** of A under f.



For $B \subseteq Y$, the set $f^{-1}(B) \subseteq X$ defined as

$$f^{-1}(B) = \{ x \in X \mid f(x) \in B \}$$

is the **inverse image** or **preimage** of B under f.

In the next lecture, we consider how preimages and images commute with unions and intersections, or not...

MATH 401: Lecture 4 (08/28/2025)

Today:
* images/preimages and unions/intersections
* injective/surjective functions
* relations, equivalence relations, partitions

We now consider how images and inverse images commute (or not) with unions and intersections of families of sets.

Prop 1.4.1 Inverse images commute with arbitrary unions and intersections:

$$f^{-1}\Big(\bigcup_{B\in\mathcal{B}} B\Big) = \bigcup_{B\in\mathcal{B}} f^{-1}(B) \quad \text{and}$$

"inverse of union = union of inverses"

$$f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big) = \bigcap_{B\in\mathcal{B}} f^{-1}(B)$$

"inverse of intersection = intersection of inverses"

Proof (of the second statement) → See LSIRA for proof of first statement

($\subseteq$) Let $x \in f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big) \Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B$

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in f^{-1}(B)$ for every $B \in \mathcal{B}$.

$\Rightarrow x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$.

($\supseteq$) Let $x \in \bigcap_{B\in\mathcal{B}} f^{-1}(B)$

$\Rightarrow x \in f^{-1}(B)$ for every $B \in \mathcal{B}$.

$\Rightarrow f(x) \in B$ for every $B \in \mathcal{B}$.

$\Rightarrow f(x) \in \bigcap_{B\in\mathcal{B}} B \Rightarrow x \in f^{-1}\Big(\bigcap_{B\in\mathcal{B}} B\Big)$. $\square$

We saw that inverse images commute with unions and intersections. But forward images behave a bit differently.

<u>Prop 1.4.2</u> $f: X \to Y$ is a function, $\mathcal{A}$ is a family of subsets of X.

Then $f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A)$, $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

<u>Proof</u>

('$\subseteq$') Let $y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$

"There exists"

$\Rightarrow \exists\, x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x)=y$.

$\Rightarrow x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x)=y$

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow y \in \bigcup_{A\in\mathcal{A}} f(A)$.

($\supseteq$) Let $y \in \bigcup_{A\in\mathcal{A}} f(A)$.

$\Rightarrow y \in f(A)$ for at least one $A \in \mathcal{A}$

$\Rightarrow \exists\, x \in A$ for at least one $A \in \mathcal{A}$ such that $f(x)=y$.

$\Rightarrow \exists\, x \in \bigcup_{A\in\mathcal{A}} A$ such that $f(x)=y$.

$\Rightarrow y \in f\left(\bigcup_{A\in\mathcal{A}} A\right)$

LSIRA gives a slightly different proof for ($\supseteq$):

"for all"

$A \subseteq \bigcup_{A\in\mathcal{A}} A \quad \forall A \in \mathcal{A}$

$\Rightarrow f(A) \subseteq f\left(\bigcup_{A\in\mathcal{A}} A\right) \quad \forall A \in \mathcal{A}$

Since this result holds for every $A \in \mathcal{A}$, we can write

$\bigcup f(A) \subseteq f\left(\bigcup_{A\in\mathcal{A}} A\right)$.

$\square$

We consider intersections now: $f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A)$.

<u>Proof for ('$\subseteq$')</u>

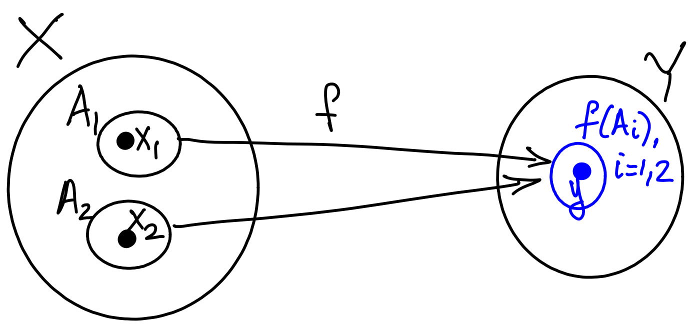
$$\bigcap_{A\in\mathcal{A}} A \subseteq A \quad \forall A\in\mathcal{A}$$

$$\Rightarrow f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq f(A) \ \forall A\in\mathcal{A}.$$

Since this inclusion holds for every $A\in\mathcal{A}$, we get

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) \subseteq \bigcap_{A\in\mathcal{A}} f(A).$$

<u>Counterexample for ('$\supseteq$') for $\cap$</u>



For $x_1 \neq x_2$, $x_1, x_2 \in X$, let $f(x_i) = y$, $i=1,2$.

Let $A_i = \{x_i\}$, $i=1,2$. $\Rightarrow \bigcap_{i=1,2} A_i = \phi$ (empty set).

But note that $f(A_i) = \{y\}$, $i=1,2$.

$\Rightarrow f\left(\bigcap_{i=1,2} A_i\right) = \phi$. But $\bigcap_{i=1,2} f(A_i) = \{y\} \neq \phi$.

$$\Rightarrow \bigcap_{i=1,2} f(A_i) \not\subseteq f\left(\bigcap_{i=1,2} A_i\right).$$

But we get this reverse inclusion if we specify that f is injective.

Def Let $f: X \to Y$ be a function.

f is **injective** (1-to-1) if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

Equivalent definition: For any $y \in Y$, there is **at most** one $x \in X$ s.t. $f(x) = y$.
→ there could be no $x \in X$

f is **surjective** (onto) if for every $y \in Y$, there is **at least** one $x \in X$ such that $f(x) = y$.
→ there could be more than one

f is **bijective** if it is both injective and surjective.

LSIRA 1.4 prob 4 (pg 17)

Let $f: \underset{X}{\mathbb{R}} \to \underset{Y}{\mathbb{R}}$ be a strictly increasing function, i.e.,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for } x_i \in \mathbb{R},\ i = 1, 2.$$

1. Show that f is injective.
2. Does it have to be surjective? → Either give a proof or a counterexample.

1. We show $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
   Without loss of generality (WLOG), let $x_1 < x_2$. → The same result holds when $x_2 < x_1$ as well.
   Then $f(x_1) < f(x_2)$, as f is strictly increasing.
   Hence $f(x_1) \neq f(x_2)$, and so f is injective.

2. No. $f = \arctan(x)$ is strictly increasing.
   $f: \mathbb{R} \to \mathbb{R}$, but $\arctan(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}$.

   So f need not be surjective.

   Another example is $f = e^x$.

Relations (LSIRA 1.5)

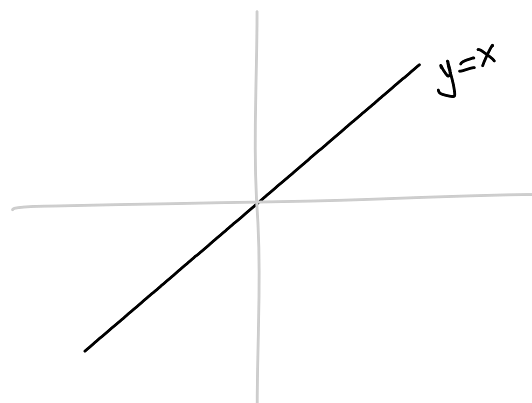
We had seen functions, where a unique $y \in Y$ is assigned for each $x \in X$ by $f: X \to Y$. But entities are related in other ways — numbers are $>$ or $<$ each other, lines are parallel, etc. We define relations formally to describe such dependencies.

Def A relation R on a set X is a subset of $X \times X$.

Cartesian product of X with itself

We write xRy, $(x,y) \in R$, or $x \sim y$.

e.g., $R = \{(x,y) \in \mathbb{R}^2 \mid x = y\}$.



Recall, $y = x$ is the 45° line through $(0,0)$. All points are "related" by them belonging to this line.

Here is another relation (on integers):

$$P = \{(x,y) \in \mathbb{Z}^2 \mid x, y \text{ have same parity}\}.$$

So, all odd integers are related, and so are all even integers.

Some relations have more structure than default — as defined below.

Equivalence Relations

Def A relation $\sim$ on X is an **equivalence relation** if it is

(i) reflexive, i.e., $x \sim x \;\forall x \in X$;

(ii) symmetric, i.e., $x \sim y \Rightarrow y \sim x \;\forall x, y \in X$; and

(iii) transitive, i.e., $x \sim y,\, y \sim z \Rightarrow x \sim z \;\forall x, y, z \in X$.

Note that $<$ is not reflexive, or symmetric; e.g., $5 \not< 5$, and $3 < 5 \not\Rightarrow 5 < 3$.

Def Given an equivalence relation $\sim$ on X, we define the **equivalence class** $[x]$ of $x \in X$ as

$$[x] = \{y \in X \mid x \sim y\}.$$

→ the set of all "relatives" of x

The collection of equivalence classes forms a <u>partition</u> of X.

Def A **partition** $\mathcal{P}$ of X is a family of nonempty subsets of X such that $x \in X$ satisfies $x \in P \in \mathcal{P}$ for exactly one P in $\mathcal{P}$ (for every $x \in X$).

The elements P of $\mathcal{P}$ are called **partition classes** of $\mathcal{P}$.

e.g., $\mathcal{P} = \{ \underbrace{\{2k, k \in \mathbb{Z}\}}_{\text{even integers}}, \underbrace{\{2k+1, k \in \mathbb{Z}\}}_{\text{odd integers}} \}$ is a partition of $\mathbb{Z}$.

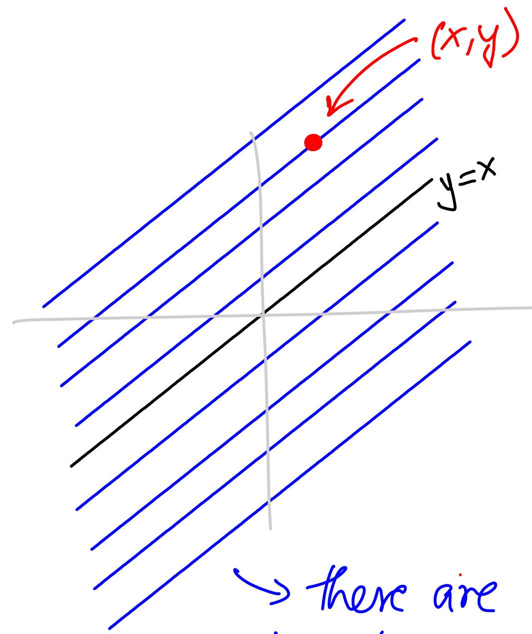
Here is a direct example of a partition of $\mathbb{R}^2$.

The collection of all lines with slope $=1$ $(45°)$ is a partition of $\mathbb{R}^2$.

Any point in $\mathbb{R}^2$ belongs to exactly one line with a slope of $m=1$ (i.e., $45°$ degree slope).

We have not checked that the defining relation is an equivalence relation, but this can be done easily.

→ recall, the point-slope form of the equation of a line: $\frac{y-y_0}{x-x_0} = m$, given slope m and one point (x_0, y_0).



→ there are infinitely many lines with slope $m=1$.

MATH 401 : Lecture 5 (09/02/2025)

Today: * equivalence relations and partitions
* countability

Recall: * $\sim$ is an equivalence relation on X : $x\sim x$, $x\sim y \Rightarrow y\sim x$, $x\sim y, y\sim z \Rightarrow x\sim z$
* partition of X $\mathcal{P}=\{P\}$

We show that equivalence relations naturally define partitions.

Prop 1.5.3 If $\sim$ is an equivalence relation on X, then the collection of equivalence classes $\mathcal{P}=\{[x] \mid x\in X\}$ is a partition of X.

Proof We show each $x\in X$ belongs to exactly one equivalence class.

$x\sim x$ $\quad$ $\sim$ is equivalence relation, so is reflexive ((i))

$\Rightarrow x\in [x]$ $\longrightarrow$ So, each $x\in X$ belongs to at least its own class.

We now show if $x\in [y]$ for $y\in X$, $y\neq x$, then $[x]=[y]$.

We show $[x]\subseteq[y]$ and $[x]\supseteq[y]$.

($\subseteq$) Let $z\in[x]$

$\Rightarrow x\sim z$ $\quad$ Definition of $[x]$

We assumed $x\in[y] \Rightarrow y\sim x$

$\sim$ is an equivalence relation, so $\underbrace{y\sim x, x\sim z \Rightarrow y\sim z}_{\sim \text{ is transitive ((iii))}}$.

$\Rightarrow z\in[y]$.

($\supseteq$) let $z\in[y]$ $\quad \Rightarrow y\sim z$

Also, $x\in[y] \Rightarrow y\sim x$

$\sim$ is equivalence relation $\Rightarrow x\sim y$ $\quad$ ($\sim$ is symmetric (ii))

$\Rightarrow x\sim y, y\sim z \Rightarrow x\sim z$ $\quad$ ($\sim$ is transitive (iii))

$\Rightarrow z\in[x]$.

$\square$

LSIRA 1.5 Prob 5 (Pg 20) Let $\sim$ be a relation on $\mathbb{R}^3$ defined as

$$(x,y,z) \sim (x',y',z') \iff 3x-y+2z = 3x'-y'+2z'.$$

Show that $\sim$ is an equivalence relation. Describe its equivalence classes.

We check that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: $(x,y,z) \sim (x,y,z)$, as $3x-y+2z = 3x-y+2z$. ✓

Symmetric: $(x,y,z) \sim (x',y',z') \Rightarrow (x',y',z') \sim (x,y,z)$ holds as

$$3x-y+2z = 3x'-y'+2z' \Rightarrow$$

$$3x'-y'+2z' = 3x-y+2z. \checkmark$$

$a=b \Rightarrow b=a$ for $a,b \in \mathbb{R}$.

Transitive: $(x,y,z) \sim (x',y',z')$ and $(x',y',z') \sim (x'',y'',z'')$

$\Rightarrow (x,y,z) \sim (x'',y'',z'')$ also holds, as

$$3x-y+2z = 3x'-y'+2z' \text{ and } 3x'-y'+2z' = 3x''-y''+2z''$$

$$\Rightarrow 3x-y+2z = 3x''-y''+2z''. \checkmark$$

$$[(x,y,z)] = \{(x',y',z') \in \mathbb{R}^3 \mid 3x-y+2z = 3x'-y'+2z'\}$$

If we set $3x-y+2z = d \in \mathbb{R}$, then

$$[(x,y,z)] = \{\underbrace{(x',y',z') \in \mathbb{R}^3 \mid 3x'-y'+2z' = d}\}$$

plane with normal vector $(3,-1,2)$ (or $\begin{bmatrix}3\\-1\\2\end{bmatrix}$) through (x,y,z).

We can describe the equivalence classes as follows.
The equivalence class of a point in $\mathbb{R}^3$ is the plane with normal $(3,-1,2)$ passing through that point.

We write $\mathbb{R}^3/\sim$ for the set of all equivalence classes of $\sim$.

Def If $\sim$ is an equivalence relation on X, then $X/\sim$ is the set of all equivalence classes under $\sim$. "X quotient $\sim$"

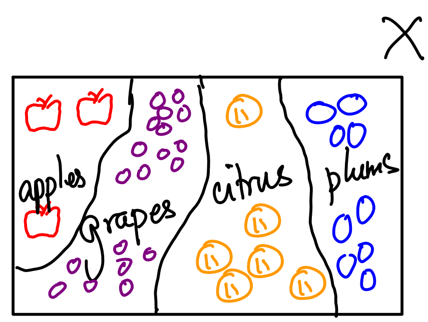
$\mathbb{R}^3/\sim$ here is the set of all planes with normal $(3,-1,2)$.

Note that any point $(x,y,z) \in \mathbb{R}^3$ belongs to exactly one plane with normal $(3,-1,2)$. Also, all such parallel planes together cover all of $\mathbb{R}^3$, i.e., $\mathbb{R}^3/\sim$ is indeed a partition of $\mathbb{R}^3$. Note the similarity to previous example of 45° lines in $\mathbb{R}^2$.

<u>Another example on equivalence classes and Partitions</u>

Let X be the set of all fruits in a grocery store. We can group them into fruit types (classes), e.g., apples, citrus, grapes, tomatoes, plums, etc. Note that apples could include honeycrisp, red delicious, etc. (varities of apples)

$\mathcal{P}$: A partition of X into fruit classes may look like this $\longrightarrow$



$$\mathcal{P} = \{\underset{P_1}{\text{apples}}, \underset{P_2}{\text{grapes}}, \underset{P_3}{\text{citrus}}, \underset{P_4}{\text{plums}}, \dots\}$$

Note that any individual fruit belongs to exactly one class. $\mathcal{P}$ is indeed a partition of X.

<u>Equivalence relation $\sim$ on X associated with $\mathcal{P}$</u>

For fruits x, y, $x \sim y$ if x and y are the same fruit type.
$\sim$ is indeed an equivalence relation (can check it's reflexive, symmetric, transitive).

What is the equivalence class $[x]$ of a fruit x?

$[x]$ is the set of all fruits of its type in the store.
e.g., x = Valencia orange, $[x] = \{$set of all citrus fruits$\}$.

What is the quotient space $X/\sim$? $X/\sim$ is the set of all fruit types.

So $X/\sim = \{$apples, citrus, $\dots\}$

Check all problems on equivalence relations from LSIRA.

LSIRA 1.6 Countability

We typically count a set of objects as $1, 2, 3, \ldots$, i.e., by numbering or indexing the first element, then the second one, etc. We can talk about sets being countable (or not) in general.

Def A set A is **countable** if it is possible to list all elements of A as $a_1, a_2, \ldots, a_n, \ldots$

set of natural numbers

e.g., $\mathbb{N}$ is countable — just list the elements as $1, 2, 3, \ldots$.

We could use a little more formal definition of a countable set, than the one given above (as listed in LSIRA).

Def A set A is countable if there exists an injective function $f : A \to \mathbb{N}$.

The function f is the "indexing" or "numbering" function that assigns a separate natural number to each element of A.

Note that finite sets are always countable — we can always list the elements in a sequence. Things are more interesting for infinite sets.

Def If f is also surjective, i.e., it is bijective, then A is **countably infinite**, i.e., it is countable and is infinite.

e.g., $\mathbb{Z}$ is countable.

We can list all integers as

$$
\begin{array}{lccccccccc}
 & 0, & 1, & -1, & 2, & -2, & 3, & -3, & \ldots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\text{index} & 1 & & 3 & & 5 & & 7 & \ldots \\
 & & 2 & & 4 & & 6 & \ldots &
\end{array}
$$

This is just one way to list all integers. Other ways could be devised as well.

Note how the indices are listed. The positive integers are the even entries in the list, and the negative integers (& 0) are the odd entries in the list

Or, we can define $f: \mathbb{Z} \to \mathbb{N}$ as

$$f(z) = \begin{cases} 2z, & z > 0 \\ 1-2z, & z \leq 0 \end{cases}$$

We can specify $f^{-1}(\cdot)$ as follows:

$$f^{-1}(n) = \begin{cases} n/2, & n \text{ even} \\ \frac{-n+1}{2}, & n \text{ odd.} \end{cases}$$

f is bijective, and hence $\mathbb{Z}$ in countably infinite.

Proposition 1.6.1 If A, B are countable, then so is $A \times B$.
↪ Cartesian product

A, B are countable $\Rightarrow \exists$ lists $\{a_n\}, \{b_n\}$ containing all elements of A and B, respectively.

$$\Rightarrow \{\underbrace{(a_1,b_1)}_{1+1}, \underbrace{(a_1,b_2)}_{\substack{1+2\\=3}}, \underbrace{(a_2,b_1)}_{\substack{2+1\\=3}}, \underbrace{(a_1,b_3)}_{\substack{1+3\\=4}}, \underbrace{(a_2,b_2)}_{\substack{2+2\\=4}}, \underbrace{(a_3,b_1)}_{\substack{3+1\\=4}}, \dots\}$$

index

is a list containing all elements of $A \times B$.

Note the index trick: we list pairs of elements (a_i, b_j) with $a_i \in \{a_n\}$ and $b_j \in \{b_n\}$ such that the sum of their indices increase as natural numbers. Thus, $i+j=2$, and then all options for $i+j=3$, followed by all options for $i+j=4$, and so on.

This index trick could be used to show other sets are countable, e.g., the cartesian product of k countable sets is countable. ($A_1 \times A_2 \times \dots \times A_k$, where A_i is countable for $1 \leq i \leq k$).

LSIRA 1.6 Prob 1 (Pg 22) Show that the subset of a countable set is countable.

Let $B \subset A$, where A is countable.

As A is countable, there is a list $a_1, a_2, \ldots, a_n, \ldots$ such that every $a_i \in A$ is included in the list.

from the list

Let $n_1 \in \mathbb{N}$ be the smallest natural number such that $a_{n_1} \in B$.

And let $n_2 \in \mathbb{N}$, $n_2 > n_1$, be the smallest number such that $a_{n_2} \in B$, and let $n_3 > n_2$, $n_3 \in \mathbb{N}$, be the smallest number such that $a_{n_3} \in B$, and so on.

We form a new list with $b_i = a_{n_i}$, $i = 1, 2, 3, \ldots$

$\Rightarrow$ $b_1, b_2, b_3, \ldots$ is a listing of all elements in B, ensuring that B is countable. $\square$

indeed, we will miss no elements of B in this process, and all of them are included in the new list.

Check Prop 1.6.2: (in LSIRA) $\bigcup_{n \in \mathbb{N}} A_n$ is countable when A_n is countable $\forall n$.

We can use a similar indexing trick as in Prop. 1.6.1.

Countability is one way to compare two infinite sets. We know $\mathbb{R} \supseteq \mathbb{Q}$, but both have infinitely many elements. Intuitively, we know $\mathbb{R}$ is bigger as it contains irrational numbers in addition to rationals.

We'll first show that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is, in fact, uncountable. More in the next lecture...

MATH401: Lecture 6 (09/04/2025)

Today: * $\mathbb{Q}$ is countable, $\mathbb{R}$ is uncountable
* ϵ-δ proofs, convergence

Recall: **Proposition 1.6.1** If A, B are countable, then so is $A \times B$.

Proposition 1.6.3 $\mathbb{Q}$ is countable.

↳ set of all rational numbers, $\frac{p}{q}$ for $p \in \mathbb{Z}, q \in \mathbb{N}$

This representation includes all negative rationals. Also, $q \in \mathbb{N}$ avoids $q=0$.

We first observe that $\mathbb{Z} \times \mathbb{N}$ is countable, as we showed that $\mathbb{Z}$ and $\mathbb{N}$ are both countable, and then applying Proposition 1.6.1.

$\Rightarrow \mathbb{Z} \times \mathbb{N}$ can be listed as, for instance,
$\{\{(a_1, b_i)\}_{i=1}^{\infty}, \{(a_2, b_i)\}_{i=1}^{\infty}, \dots, \{(a_k, b_i)\}_{i=1}^{\infty}, \dots\}$ where $\{a_n\}$ and $\{b_n\}$ are listings for $\mathbb{Z}$ and $\mathbb{N}$, respectively.

But $\left\{\left\{\frac{a_1}{b_i}\right\}_{i=1}^{\infty}, \left\{\frac{a_2}{b_i}\right\}_{i=1}^{\infty}, \dots, \left\{\frac{a_k}{b_i}\right\}_{i=1}^{\infty}, \dots\right\}$ is a listing of $\mathbb{Q}$. $\square$

Let's consider any rational number, e.g., $\frac{2}{5}$.
How many times does $\frac{2}{5}$ appear in this listing? Once, exactly as $\frac{2}{5}$.

But infinitely many times as a value, because $\frac{2}{5} = \frac{4}{10} = \frac{20}{50} = \dots$

In fact, every rational number appears infinitely many times in this list. But that is not a problem for countability.

We now show that the set of all reals is uncountable.

Theorem 1.6.4 $\mathbb{R}$ is uncountable.

Consider $[0,1] \subset \mathbb{R}$. We show that $[0,1]$ is uncountable. To get a contradiction, assume that $[0,1]$ is countable.

As there are infinitely many real #s between 0 and 1. $[0,1]$ is a countably infinite set (under assumption).

We can list all these real numbers as follows:

Note that each number has infinitely many decimal digits (they could be all zeros after some number of places)

$$\begin{aligned} r_1 &= 0.a_{11}a_{12}a_{13}\cdots \\ r_2 &= 0.a_{21}a_{22}a_{23}\cdots \\ r_3 &= 0.a_{31}a_{32}a_{33}\cdots \\ &\vdots \end{aligned}$$

a_{ij} = j^{th} decimal digit in the i^{th} real number (in the list).

$a_{ij} \in \{0,1,2,\ldots,9\}$.

We create a new real number in $[0,1]$ as follows.

$$s = 0.d_1d_2d_3\cdots \quad \text{where}$$

$$d_i = \begin{cases} 1 & \text{if } a_{ii} \neq 1, \text{ and} \\ 2 & \text{if } a_{ii} = 1. \end{cases}$$

e.g.,

$$\begin{aligned} r_1 &= 0.02534\ldots \\ r_2 &= 0.8076\ldots \\ r_3 &= 0.30914\ldots \\ r_4 &= 0.00207\ldots \\ &\vdots \end{aligned}$$

Then $s = 0.1211\ldots$

Note that s has infinitely many decimal digits.

So, s is different from r_i for each i.
This contradicts the assumption that $\{r_i\}$ contains **every** real number in $[0,1]$. Hence $[0,1]$ is uncountable.

Since $\mathbb{R} \supset [0,1]$, and $[0,1]$ is uncountable, $\mathbb{R}$ is also uncountable. $\square$

This is a standard trick we use to show a set is uncountable. We assume it is countable, and start with a listing. Then we identify an element that is distinct from every element in the listing, contradicting the assumption that the listing includes all such elements.

Corollary. The set of irrational numbers is uncountable.

We showed $\mathbb{Q}$ is countable, and $\mathbb{R}$ is uncountable.
The set of irrationals $= \mathbb{R}/\mathbb{Q}$ is hence uncountable.

LSIRA Chapter 2 Foundations of Calculus

2.1. ϵ-δ Definitions and Proofs

Norms and Distances

Euclidean distance, by default

Def The **distance** between $\bar{x} = (x_1, \dots, x_m)$ $\left(\text{or } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right)$ and $\bar{y} = (y_1, \dots, y_m)$, two points in $\mathbb{R}^m$ is

$$\|\bar{x} - \bar{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_m - y_m)^2}.$$

My notation: $\bar{x}, \bar{y}, \bar{\alpha}, \bar{\theta}$, etc. are vectors → lower case letters with a bar.

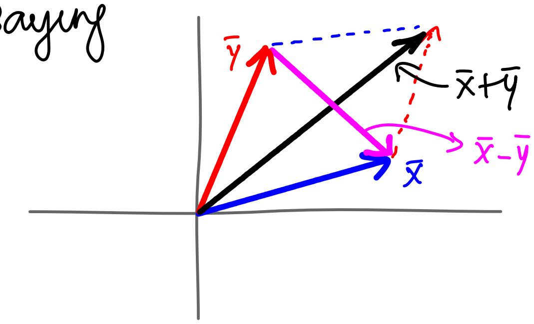
For $m = 1$, $\|x - y\| = \sqrt{(x-y)^2} = |x - y|$ → absolute value of $x - y$

think of it as just the distance between two points in $\mathbb{R}$.

Triangle Inequality

$$\forall\, \bar{x}, \bar{y} \in \mathbb{R}^m, \quad \|\bar{x} + \bar{y}\| \le \|\bar{x}\| + \|\bar{y}\|.$$

We could interpret the triangle inequality as saying length of diagonal $\le$ sum of lengths of sides of the parallelogram.

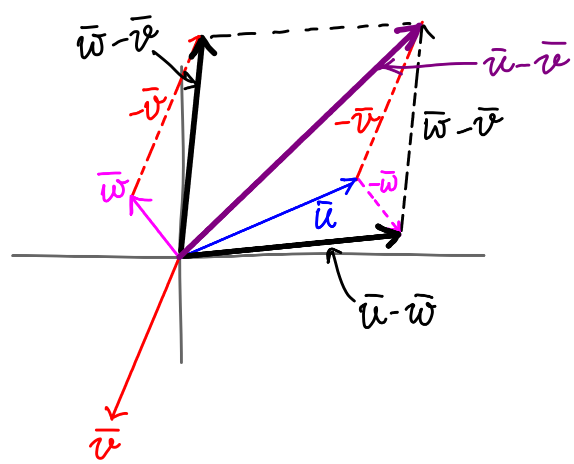


With $\bar{x} = \bar{u} - \bar{w}$, $\bar{y} = \bar{w} - \bar{v}$, we get

$$\|\bar{u} - \bar{v}\| = \|\bar{u} - \bar{w} + \bar{w} - \bar{v}\| \le \|\bar{u} - \bar{w}\| + \|\bar{w} - \bar{v}\|$$

for $\bar{u}, \bar{v}, \bar{w} \in \mathbb{R}^m$

Illustration of the above version in 2D: notice the parallelogram here as well!

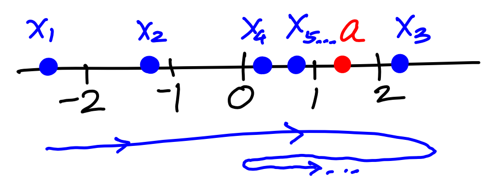


Convergence of Sequences

As a first use of distances, we consider convergence of sequences. How do we say a sequence $\{x_n\}$ converges to a real number a? We should be able to get arbitrarily close to a by going far enough (large n) into the sequence.

Def 2.1.1 A sequence $\{x_n\}$ of real numbers converges to $a \in \mathbb{R}$ if for every $\epsilon > 0$ (no matter how small), there exists an $N \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ for all $n \geq N$. We write $\lim_{n\to\infty} x_n = a$.

Here is a pictorial representation of the convergence, with the "path" drawn separately below for clarity.



LSIRA 2.1 Prob 1 (Pg 29)

Show that if $\{x_n\}$ converges to a, then the sequence $\{Mx_n\}$ converges to Ma. Use the definition of convergence to explain how you choose N.

Given $\{x_n\} \to a$ $\left(\lim_{n\to\infty} x_n = a\right)$ $\Rightarrow$ $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - a| < \epsilon \ \forall n \geq N$.

We want to show $\{Mx_n\} \to Ma$. We want to show that $\forall \epsilon > 0, \exists N' \in \mathbb{N}$ s.t. $|Mx_n - Ma| < \epsilon \ \forall n \geq N$.

Note that when $M=0$, the result holds trivially, as $Mx_n = 0 \ \forall n$, and $Ma = 0$. Hence $|Mx_n - Ma| = 0 < \epsilon$ for any $\epsilon > 0$ for $n \geq \underset{N'}{1}$.

Also note that both $M > 0$ and $M < 0$ are possible.

Let's assume $M \neq 0$.

6.6

First, observe that $|Mx_n - Ma| = |M(x_n - a)| = |M||x_n - a|$.

Note that when $|x_n - a| < \epsilon' = \dfrac{\epsilon}{|M|}$, $\quad |M||x_n - a| < \epsilon$.

But since $\{x_n\} \to a$, given $\epsilon' = \dfrac{\epsilon}{|M|} > 0$, $\exists\, N' \in \mathbb{N}$ s.t. $|x_n - a| < \epsilon'$ for all $n \geq N'$. We can choose $N = N'$, and get

$$|x_n - a| < \epsilon' = \frac{\epsilon}{|M|} \quad \forall n \geq N'$$

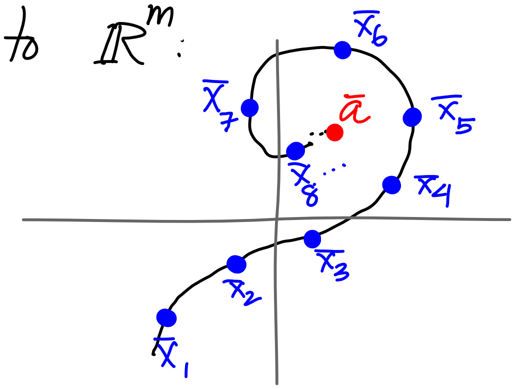
$$\Rightarrow |M||x_n - a| = |Mx_n - Ma| < \epsilon \quad \forall\, n \geq N'$$

$\Rightarrow \{Mx_n\}$ converges to Ma. $\square$

MATH 401: Lecture 7 (09/09/2025)

Today: * convergence in $\mathbb{R}^m$
* continuity of functions

We extend the notion of convergence in $\mathbb{R}$ to $\mathbb{R}^m$:



The definition naturally extends to $\mathbb{R}^m$ once we think of $|x_n - a|$ as the distance between x_n and a.

Def 2.1.2 A sequence $\{\bar{x}_n\}$ of points in $\mathbb{R}^m$ converges to $\bar{a} \in \mathbb{R}^m$ if $\forall \epsilon > 0$, $\exists$ an $N \in \mathbb{N}$ such that $\|\bar{x}_n - \bar{a}\| < \epsilon$ $\forall n \geq N$. We write $\lim_{n \to \infty} \bar{x}_n = \bar{a}$.

LSIRA Prob 2.1.3 $\{\bar{x}_n\}$, $\{\bar{y}_n\}$ are two sequences in $\mathbb{R}^m$ where $\{x_n\} \xrightarrow{\text{converges}} \bar{a}$, and $\{\bar{y}_n\} \to \bar{b}$. Then show that $\{\bar{x}_n + \bar{y}_n\}$ converges to $\bar{a} + \bar{b}$.

We want to show: $\forall \epsilon > 0$ (any given ϵ), $\exists N \in \mathbb{N}$ such that

$$\|(\bar{x}_n + \bar{y}_n) - (\bar{a}_n + \bar{b}_n)\| < \epsilon \quad \forall n \geq N.$$

same ϵ as our target

Hint, hint, hint! $\|\bar{x} + \bar{y} + \bar{z}\| \leq \|\bar{x}\| + \|\bar{y}\| + \|\bar{z}\|$ by applying triangle inequality twice. We often choose $\epsilon/3$ (instead of $\frac{\epsilon}{2}$) with 3 terms!

We are given $\{\bar{x}_n\} \to \bar{a}$, $\{\bar{y}_n\} \to \bar{b}$, so

$\exists N_1 \in \mathbb{N}$ s.t. $\|\bar{x}_n - \bar{a}\| < \frac{\epsilon}{2}$ $\forall n \geq N_1$ and

$\exists N_2 \in \mathbb{N}$ s.t. $\|\bar{y}_n - \bar{b}\| < \frac{\epsilon}{2}$ $\forall n \geq N_2$.

$\Rightarrow$ for $N = \max\{N_1, N_2\}$, we get

$$
\begin{aligned}
\|(\bar{x}_n + \bar{y}_n) - (\bar{a} + \bar{b})\| &= \|(\bar{x}_n - \bar{a}) + (\bar{y}_n - \bar{b})\| \\
&\leq \|\bar{x}_n - \bar{a}\| + \|\bar{y}_n - \bar{b}\| && \text{by triangle inequality} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && \text{as } N \geq N_1, N \geq N_2.
\end{aligned}
$$

$\Rightarrow \{\bar{x}_n + \bar{y}_n\} \to \bar{a} + \bar{b}$. $\square$

Continuity

$f: \mathbb{R} \to \mathbb{R}$. When is f continuous at $x = a$?

For sequences $\{x_n\} \to a$, we go "far enough out", i.e., $\forall n \geq N \in \mathbb{N}$. Instead of $N \in \mathbb{N}$, here we say $\exists\, \delta > 0$ such that if $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$ (for any given $\epsilon > 0$). In other words, $f(x)$ gets close enough to $f(a)$ whenever x is close enough to a!

Def 2.1.4 The function $f: \mathbb{R} \to \mathbb{R}$ is **continuous** at $a \in \mathbb{R}$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Equivalently, if $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

We naturally extend the definition to $\mathbb{R}^m$ using distances/norms.

LSIRA uses $\mathbf{F}$ (bold upper case F)

Def 2.1.7 The function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\bar{a} \in \mathbb{R}^n$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $\|\bar{f}(\bar{x}) - \bar{f}(\bar{a})\| < \epsilon$ whenever $\|\bar{x} - \bar{a}\| < \delta$.

By restricting our attention to a subset A of $\mathbb{R}^n$, we naturally extend the above definition to subsets of interest.

Def 2.1.8 Let $A \subset \mathbb{R}^n$, and $\bar{a} \in A$.

The function $\bar{f}: \mathbb{R}^n \to \mathbb{R}^m$ is **continuous** at $\bar{a} \in A$ if $\forall \epsilon > 0$ (no matter how small), $\exists$ a $\delta > 0$ such that $\|\bar{f}(\bar{x}) - \bar{f}(\bar{a})\| < \epsilon$ whenever $\|\bar{x} - \bar{a}\| < \delta$ and $\bar{x} \in A$.

<u>LSIRA Section 2.1 Prob 4 (extension)</u>: If $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2,3$ are all continuous at $a \in \mathbb{R}$, the show that so is $f_1 + f_2 - f_3$. (i.e., show $f_1(x) + f_2(x) - f_3(x)$ is continuous at $x = a$).

Prob 4 considers $f+g$ for two functions f, g.

Let $g(x) = f_1(x) + f_2(x) - f_3(x)$. We want to show that

$\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|g(x) - g(a)| < \epsilon$ whenever $|x-a| < \delta$.

We know: since $f_i(x)$ are continuous at $x = a$,

$\exists \delta_i > 0$ s.t. $|f_i(x) - f_i(a)| < \frac{\epsilon}{3}$ whenever $|x - a| < \delta_i$, $i = 1,2,3$.

Let $\delta = \min\limits_{i=1,2,3} \{\delta_i\}$. Then

We want x to as close to a as required in each case!

e.g., if $\delta_1 = 0.1$, $\delta_2 = 0.05$ and $\delta_3 = 0.08$, then $\delta \le 0.05$ works!

$$
\begin{aligned}
|g(x) - g(a)| &= \left|\left(f_1(x) + f_2(x) - f_3(x)\right) - \left(f_1(a) + f_2(a) - f_3(a)\right)\right| \\
&= \left|\left(f_1(x) - f_1(a)\right) + \left(f_2(x) - f_2(a)\right) + \left(f_3(a) - f_3(x)\right)\right| \\
&\le |f_1(x) - f_1(a)| + |f_2(x) - f_2(a)| + \underbrace{|f_3(a) - f_3(x)|}_{= |f_3(x) - f_3(a)|}
\end{aligned}
$$

by triangle inequality (applied twice)

$$
\begin{aligned}
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \text{as } \delta \le \delta_i \text{ for } i = 1,2,3 \\
&= \epsilon \quad \text{whenever } |x - a| < \delta.
\end{aligned}
$$

□

LSIRA Proposition 2.1.9 Let $g:\mathbb{R}\to\mathbb{R}$ be continuous at $a\in\mathbb{R}$, and $g(a)\neq 0$.
Show that $h(x)=\dfrac{1}{g(x)}$ is continuous at $x=a$.

Need to show: $\forall\, \epsilon>0,\ \exists\, \delta>0$ s.t. $|h(x)-h(a)|<\epsilon$
whenever $|x-a|<\delta$.

We want to show that

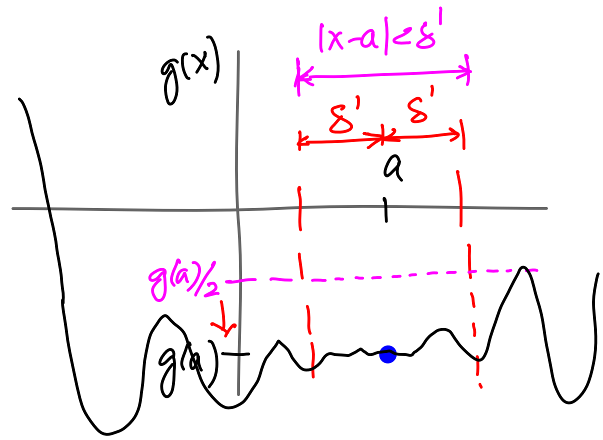
$$|h(x)-h(a)| = \left|\frac{1}{g(x)}-\frac{1}{g(a)}\right| < \epsilon.$$

$$\left|\frac{1}{g(x)}-\frac{1}{g(a)}\right| = \left|\frac{g(a)-g(x)}{g(x)\,g(a)}\right| = \frac{|g(x)-g(a)|}{|g(x)|\,|g(a)|}$$

$\neq 0$

We want to show that $|g(x)|$ is not too small. Else, the fraction could be too large.

There must exist some $\delta'>0$ such that $|g(x)|>\dfrac{|g(a)|}{2}$ whenever $|x-a|<\delta'$, as $g(a)\neq 0$.



In the picture here, notice that $g(x)$ lies "below" the $\frac{g(a)}{2}$ level, i.e., far enough away from zero, when $|x-a|<\delta'$.

Also, $g(x)$ is continuous at $x=a \implies$
$\exists\, \delta''>0$ s.t. $|g(x)-g(a)|<\epsilon'$ whenever $|x-a|<\delta''$.

Pick $\delta = \min\{\delta', \delta''\}$. Then we get that

$$\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| = \frac{|g(x)-g(a)|}{|g(x)|\,|g(a)|} < \frac{\epsilon'}{\frac{|g(a)|}{2}\,|g(a)|} = \frac{2\epsilon'}{|g(a)|^2}$$

whenever $|x-a|<\delta$.

If we choose $\epsilon' = \frac{|g(a)|^2}{2}\epsilon$, so that $\frac{2\epsilon'}{|g(a)|^2} = \epsilon$,

we get that $\left|\frac{1}{g(x)} - \frac{1}{g(a)}\right| < \epsilon$ whenever $|x-a|<\delta$.

Hence $\frac{1}{g(x)}$ is continuous at $x=a$

$\square$

In the next section, we consider the setting where the target or candidate limit (a) is not given to us.
Can we still conclude that $\{\bar{x}_n\}$ converges? When?

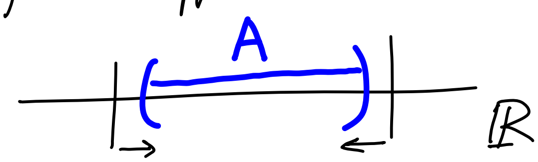
MATH 401: Lecture 8 (09/11/2025)

Today: * completeness
* sup, inf, lim sup, lim inf

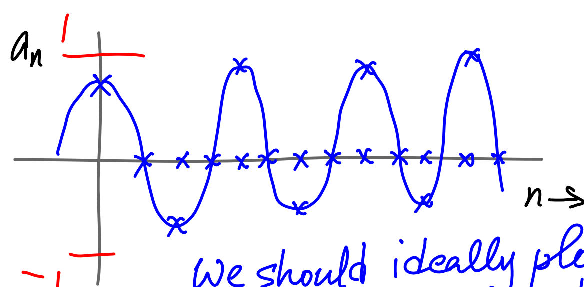
Completeness (LSIRA 2.3)

If we don't know the limit target $\bar{a}$, can we still say $\{\bar{a}_n\}$ converges? If $\{\bar{a}_n\}$ "behaves nicely" and $\bar{a}_n$'s are in a "nice space", then yes!

Here is an intuition for what we mean by "nice space". Suppose $a_n \in A$ where A is a "finite" interval (open or closed). Then we can be sure that the a_n's cannot become arbitrarily large or arbitrarily small.

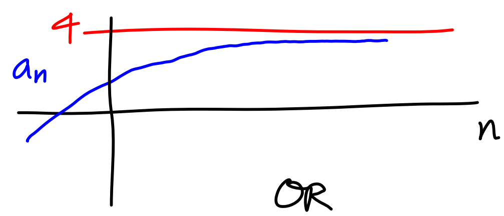


But in this example, the a_n's belong to a bounded interval $[-1,1]$, but they are not "behaving nicely" as the values oscillate between 1 and -1.

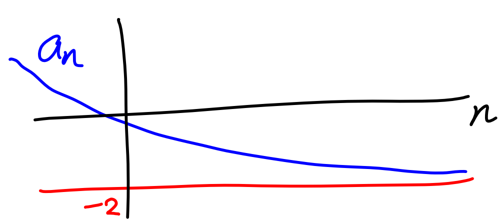


We should ideally plot the discrete pts (x), instead of the continuous curve

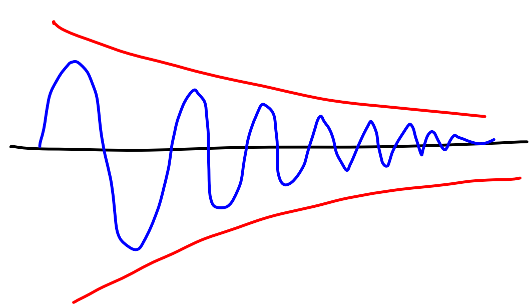
But if the a_n's are increasing and are bounded from above, or decreasing and bounded from below, we can conclude that $\{a_n\}$ converges!



OR



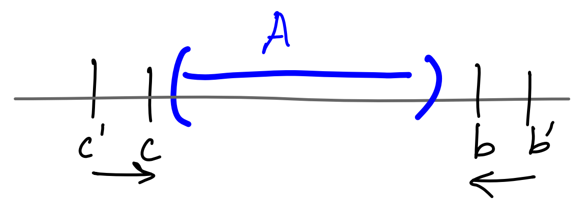
Finally, even if a_n's are oscillating, and hence not increasing/decreasing, it could still be nice if the oscillations become smaller and smaller — as shown here. Intuitively, we want the upper and lower "envelopes" to get closer and closer.



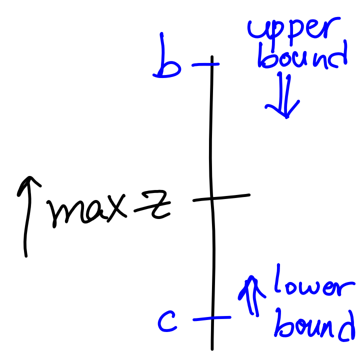
We formalize these intuitive notions of "nice" space and "nice" behavior.

Def A nonempty set $A \subset \mathbb{R}$ is **bounded above** if there exists $b \in \mathbb{R}$ such that $a \leq b \;\forall a \in A$, and is **bounded below** if there exists $c \in \mathbb{R}$ such that $a \geq c \;\forall a \in A$. We refer to b as an **upper bound**, and c as a **lower bound**.

If b is an upper bound, then any $b' > b$ is also an upper bound. Similarly, and $c' < c$ is also a lower bound.



We usually want to find a smallest upper bound, and a largest lower bound. This idea is ubiquitous in optimization, where finding the correct maximum value for a function $z = f(\bar{x})$ may be hard, but it may be easier to obtain lower/upper bounds. In order to get as best a handle on the actual max z value, we try to find the smallest upper bound, and the biggest lower bound that work.



In the same way, we want to "estimate" A as accurately as possible by finding the smallest upper bound and the largest lower bound for the set.

The Completeness Principle

Every nonempty subset A of $\mathbb{R}$ that is bounded above has a least upper bound. This bound is called the **supremum of A**, written $\sup A$.

Similarly every nonempty subset A of $\mathbb{R}$ that is bounded below has a greatest lower bound, called the **infimum of A**, written $\inf A$.

LSIRA 2.2 Problem 1 Argue that $\sup [0,1) = 1$ and $\sup [0,1] = 1$.

Let $A = [0,1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$. So $x \in A$ can be arbitrarily close to 1, i.e., $x = 1-\epsilon$, $\epsilon > 0$, arbitrarily small. Hence any $1-\epsilon$ cannot be an upper bound for A, since $\forall \epsilon > 0$, $\exists\, 1-\epsilon' \in A$ s.t. $1-\epsilon' > 1-\epsilon$.

$\Rightarrow$ $b \geq 1$ satisfies $x \leq b$ $\forall x \in A$, and hence $\sup A = 1$.

The same argument holds for $[0,1]$ too. Note that the sup is in A in the latter case, but $\sup A \notin A$ for $A = [0,1)$.

So, what is the big deal about the completeness principle?

First, it does not hold over $\mathbb{Q}$ (rationals), as, e.g.,

$A = \{x \in \mathbb{R} \mid x^2 < 3\}$ has $\sup A = \sqrt{3}$. But

$B = \{x \in \mathbb{Q} \mid x^2 < 3\}$ has no supremum in $\mathbb{Q}$!

$\rightarrow$ $\sqrt{3}$ is irrational, and we can get arbitrarily close to $\sqrt{3}$ using rational numbers!

We say that $\mathbb{Q}$ does not satisfy completeness principle.

Monotone Sequences, lim sup, lim inf

We now describe sequences that behave "nicely" like the bounded sets introduced earlier. We then consider how to handle sequences that are not as "nice".

"nondecreasing" if you want $a_{n+1} > a_n$ to mean "increasing"

Def A sequence $\{a_n\}$ in $\mathbb{R}$ is **increasing** if $a_{n+1} \geq a_n \; \forall n$.

A sequence $\{a_n\}$ in $\mathbb{R}$ is **decreasing** if $a_{n+1} \leq a_n \; \forall n$.

$\{a_n\}$ is **monotone** if it is either increasing or decreasing.

$\{a_n\}$ is **bounded** if $\exists \, M \in \mathbb{R}$ s.t. $|a_n| \leq M \; \forall n$.

LSIRA Theorem 2.2.2 Every monotone bounded sequence in $\mathbb{R}$ converges to a number in $\mathbb{R}$. → we do not specify which number!

Proof (for increasing sequences). We proceed in two steps.

1. $\{a_n\}$ is bounded $\Rightarrow$ $A = \{a_1, a_2, \ldots, a_n, \ldots\}$ is bounded. → set

   $\Rightarrow \exists \, a \in \mathbb{R}$ such that $\sup A = a$. → using completeness of $\mathbb{R}$

2. a is the least upper bound. → We show $\{a_n\} \to a$

   $\Rightarrow a - \epsilon$ is not an upper bound for any $\epsilon > 0$.

   $\{a_n\}$ is increasing $\Rightarrow$ $a - \epsilon < a_n \leq a \;\; \forall n \geq N$ for some N.

   $\Rightarrow |a - a_n| < \epsilon \;\; \forall \, n \geq N$, i.e., $\{a_n\}$ converges.

   $a_n - a > -\epsilon$ and $a - a_n < \epsilon$ $\square$

But what if $\{a_n\}$ is not monotone and/or not bounded? Can we still say something about $\{a_n\}$ as $n \to \infty$?

Given a general sequence $\{a_n\}$, we define two related sequences that are monotone themselves.

Def Given $\{a_k\}$, $a_k \in \mathbb{R}$, we define two new sequences $\{M_n\}$ and $\{m_n\}$ as follows.

$$M_n = \sup \{a_k \mid k \geq n\} \quad \text{and}$$

$$m_n = \inf \{a_k \mid k \geq n\}.$$

$M_n = \infty$, $m_n = -\infty$ are allowed here.

M_n "captures" how large $\{a_k\}$ can be "after" n, and m_n captures how small $\{a_k\}$ can be "after" n.

Note that $\{M_n\}$ and $\{m_n\}$ are monotone!

$\{M_n\}$ is decreasing, as suprema are taken over smaller subsets. and $\{m_n\}$ is increasing, as infima are taken over smaller subsets.

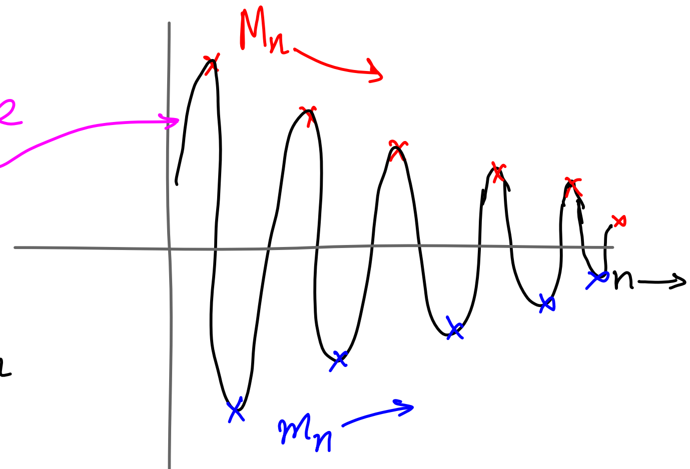
e.g., consider $A = \{1, 2, \ldots, 10\}$. The largest number in A cannot be bigger than the largest number in $A' = \{1, 2, \ldots, 7\}$, or in any $A' \subset A$, in general.

$$\Rightarrow \lim_{n \to \infty} M_n \text{ and } \lim_{n \to \infty} m_n \text{ exist!}$$

Def The **limit superior** or lim sup of the original sequence $\{a_n\}$ is $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n$.

The **limit inferior** of $\{a_n\}$ is $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n$.

We ideally want to draw a sequence of points "......" in place of the continuous curve here



It appears while $\{x_n\}$ may be "oscillating" the upper bounds M_n and lower bounds m_n appear to be converging. Hence, $\{a_n\}$ also appears to converge!

But we could have $\{a_n\}$ oscillate forever, even when M_n and m_n are finite $\forall n \geq N$.

LSIRA 2.2 Problem 4

Let $a_n = (-1)^n$. What is $\limsup_{n\to\infty} a_n$? $\liminf_{n\to\infty} a_n = ?$

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} M_n = 1.$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} m_n = -1.$$

Note than $a_n = 1 \;\forall\, n = 2k$, and $a_n = -1 \;\forall\, n = 2l+1$.

Hence $a_n \leq 1 \;\forall\, n$, and $a_n \geq -1 \;\forall\, n$.

In fact, $\{M_n\}$ and $\{m_n\}$ behave identical to $\{a_n\}$ here!

In the above problem, even though lim sup and lim inf are both finite, they are not equal, and we cannot say anything about $\{a_n\}$ converging to a limit. But when the lim sup and lim inf are equal, we get the picture drawn earlier, with $\{a_n\}$ converging to that value!

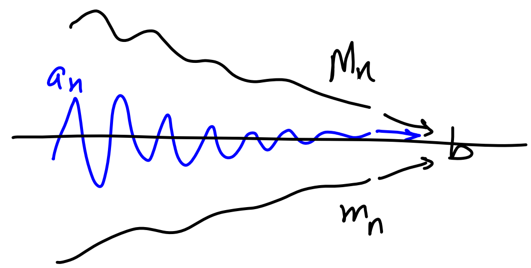
LSIRA Proposition 2.2.3 Let $\{a_n\}$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n = b$ if and only if

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b.$$

b can $\pm\infty$ here!

($\Leftarrow$) Assume $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b$

$$\Rightarrow \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = b$$



Also, $m_n \le a_n \le M_n \quad \forall n$

$$\Rightarrow \lim_{n\to\infty} a_n = b.$$

(by "squeeze law" or "squeeze theorem"; LSIRA 2.2 Problem 2 — assigned in HW4!)

We'll finish the proof in the next lecture..

MATH 401: Lecture 9 (09/16/2025)

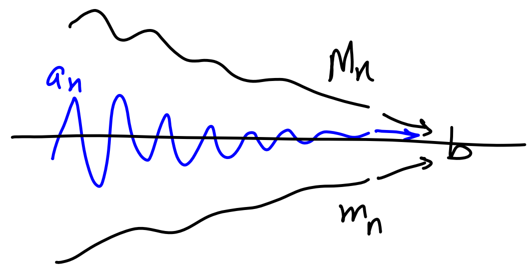
Today: * Cauchy sequences
* Intermediate value theorem (IVT)

We first present the proof of Proposition 2.2.3...

LSIRA Proposition 2.2.3 Let $\{a_n\}$ be a sequence of real numbers. Then $\lim\limits_{n\to\infty} a_n = b$ iff $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$.

b can be $\pm\infty$ here!

($\Leftarrow$) Assume $\limsup\limits_{n\to\infty} a_n = \liminf\limits_{n\to\infty} a_n = b$



$$\Rightarrow \lim_{n\to\infty} M_n = \lim_{n\to\infty} m_n = b$$

Also, $m_n \leq a_n \leq M_n \quad \forall n$

$$\Rightarrow \lim_{n\to\infty} a_n = b.$$

(by "squeeze law" or "squeeze theorem"; LSIRA 2.2 Problem 2 — assigned in HW4!)

($\Rightarrow$) Assume $\lim\limits_{n\to\infty} a_n = b$, and $b \in \mathbb{R}$.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - b| < \epsilon \quad \forall n \geq N$.

($\Leftrightarrow$ $a_n - b < \epsilon$ and $b - a_n < \epsilon$)

$|x| < 5 \Rightarrow -x < 5$ and $x < 5$

$\Rightarrow b - \epsilon < a_n < b + \epsilon \quad \forall n \geq N$

$\Rightarrow b - \epsilon < m_n < b + \epsilon$ and
$b - \epsilon < M_n < b + \epsilon \quad \forall n \geq N$

We will repeatedly use this trick of splitting $|x-y| < \epsilon$ into $x - y < \epsilon$ and $y - x < \epsilon$

Since the result holds for any $\epsilon > 0$,

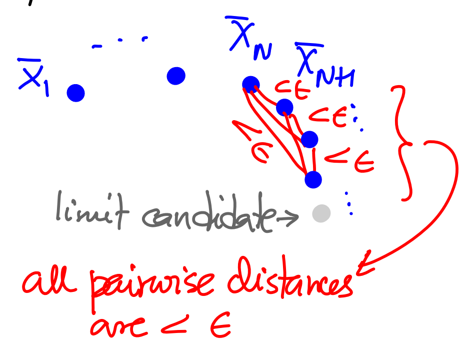
$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = b. \qquad \square$$

Cauchy Sequences

We extend the idea of completeness in $\mathbb{R}$ to $\mathbb{R}^m$. But there is no natural way to order points in $\mathbb{R}^m$ (as in $\mathbb{R}$). Instead, we say the points get closer and closer to each other.

Def 2.2.4 A sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$ is called a **Cauchy sequence** if $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s.t. $\|\bar{x}_n - \bar{x}_k\| < \epsilon \;\; \forall n, k \geq N$.

n, k are two indices, and represent any two points that are both far out enough into the sequence $(n, k \geq N)$



Completeness Result in $\mathbb{R}^m$

Theorem 2.2.5 The sequence $\{\bar{x}_n\}$ in $\mathbb{R}^m$ converges **iff** it is cauchy.

This is an iff result. We prove both directions, but one of them is easier than the other. We show the easy direction in $\mathbb{R}^m$, but the reverse direction in $\mathbb{R}$ (and can be extended to $\mathbb{R}^m$).

Proposition 2.2.6 All convergent sequences in $\mathbb{R}^m$ are cauchy.

Proof Let $\{\bar{a}_n\}$ converge to $\bar{a}$ in $\mathbb{R}^m$.

We want to show $\|\bar{a}_n - \bar{a}_k\| < \epsilon \;\; \forall n, k \geq N$ for some $N \in \mathbb{N}$.

$\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|\bar{a}_n - \bar{a}\| < \frac{\epsilon}{2}, \; \forall n \geq N$.

Ideally, we use ϵ' here, and then choose $\epsilon' = \frac{\epsilon}{2}$.

$\Rightarrow$ If $n, k \geq N$, then

$$\|\bar{a}_n - \bar{a}_k\| = \|\bar{a}_n - \bar{a} + \bar{a} - \bar{a}_k\| \leq$$

triangle inequality

$$\|\bar{a}_n - \bar{a}\| + \|\bar{a} - \bar{a}_k\| < \underbrace{\frac{\epsilon}{2} + \frac{\epsilon}{2}} = \epsilon \quad \forall n \geq N.$$

now we see why we chose $\frac{\epsilon}{2}$!

$\Rightarrow \{\bar{a}_n\}$ is cauchy. $\square$

We present proof for the reverse direction in $\mathbb{R}$. We can repeat this argument for each dimension to prove the result in $\mathbb{R}^m$. We need a lemma first.

Lemma 2.2.7 Every Cauchy sequence $\{a_n\}$ in $\mathbb{R}$ is bounded.

Want to show: $|a_n| \leq M$ for some $M \in \mathbb{R}$. note, $M \geq 0$

$\{a_n\}$ is cauchy $\Rightarrow |a_n - a_k| < \epsilon \;\; \forall n,k \geq N \in \mathbb{N}$ for any $\epsilon > 0$.

$\Rightarrow |a_n - a_N| < 1$ (for $\epsilon = 1$) the definition applies for any ϵ, so we choose $\epsilon = 1$. After all, we just need to find a valid bound

$\Rightarrow a_n - a_N < 1$ and $a_N - a_n < 1$

$\Rightarrow a_n < a_N + 1$ and $a_n > a_N - 1 \;\; \forall n \geq N$.

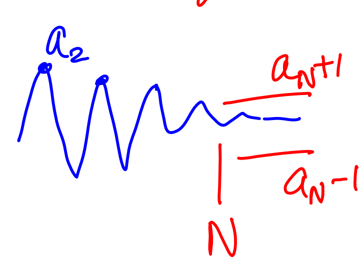
$\Rightarrow M = \max\{a_1, a_2, \ldots, a_{N-1}, a_N + 1\}$ is an upper bound, and

$m = \min\{a_1, a_2, \ldots, a_{N-1}, a_N - 1\}$ is a lower bound. $\square$

Could also get $|a_n| - |a_N| \leq |a_n - a_N| < 1$

$\Rightarrow |a_n| \leq |a_N| + 1$.

We could have a larger number among $a_1, a_2, \ldots, a_{N-1}$, which are not considered earlier since the Cauchy definition stipulates $n, k \geq N$.



Proposition 2.2.8 All Cauchy sequences in $\mathbb{R}$ converge.

<u>Proof</u> $\{a_n\}$ is <u>cauchy</u> $\Rightarrow$ $\{a_n\}$ is bounded (by Lemma 2.2.7).

$\Rightarrow M = \limsup_{n\to\infty} a_n$ and $m = \liminf_{n\to\infty} a_n$ are both finite.

We can use Proposition 2.2.3 now, if we can show $M = m$.

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|a_n - a_k| < \epsilon \ \forall n, k \geq N$.

In particular, $|a_n - a_N| < \epsilon \ \forall n \geq N$. (taking $k = N$)

$\Rightarrow \quad a_n - a_N < \epsilon$ and $a_N - a_n < \epsilon \ \forall n \geq N$

$\Rightarrow \quad a_n < a_N + \epsilon$ and $a_n > a_N - \epsilon$

i.e., $a_N - \epsilon < a_n < a_N + \epsilon \ \forall n \geq N$ holds for any $\epsilon > 0$.

$\Rightarrow \quad M_n = \sup\{a_k \mid k \geq n\} < a_N + \epsilon$ ADD

$-(m_n = \inf\{a_k \mid k \geq n\} > a_N - \epsilon) \Rightarrow -m_n < -a_N + \epsilon$

$\Rightarrow \quad M_n - m_n < 2\epsilon \ \forall n \geq N$ and for any $\epsilon > 0$, arbitrary.

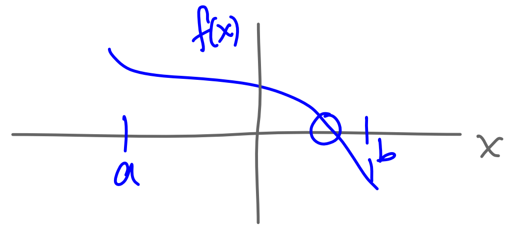
$\Rightarrow \quad M = m$ (as $n \to \infty$).

$\square$

We now present four fundamental theorems, the proofs of which use many of the results we have presented. These theorems are quite fundamental in analysis, and also finds use in many applied domains as well.

Intermediate Value Theorem

This is a rather straightforward result to understand — if a function goes from above the x-axis to below it, and is continuous, then it must cross the x-axis.



Theorem 2.3.1 (Intermediate Value Theorem) Assume $f:[a,b]\to\mathbb{R}$ is continuous, and $f(a)$ and $f(b)$ have opposite signs. Then there exists $c\in(a,b)$ such that $f(c)=0$.

We will use a characterization of continuity using sequences in the proof (from LSIRA 2.1, actually!).

Proposition 2.1.5 $f:\mathbb{R}\to\mathbb{R}$ is continuous at $x=a$ iff $\lim_{n\to\infty} f(x_n)=f(a)$ for all sequences $\{x_n\}$ that converge to a.

Proof

($\Rightarrow$) Assume f is continuous at $x=a$.

Consider $\{x_n\}\to a$, i.e., $\lim_{n\to\infty} x_n = a$.

Need to show: $\forall \epsilon>0, \exists N\in\mathbb{N}$ s.t. $|f(x_n)-f(a)|<\epsilon \;\forall n\geq N$.

$\Rightarrow \exists\, \delta>0$ s.t. $|f(x)-f(a)|<\epsilon$ whenever $|x-a|<\delta$. same

$\exists N'\in\mathbb{N}$ s.t. $|x_n-a|<\delta$ whenever $n\geq N'$. $\longrightarrow$ δ plays the "role of ϵ," i.e., the convergence definition must hold for any $\epsilon>0$, and here we choose $\epsilon=\delta$.

$\Rightarrow$ If $n\geq N'$, then $|f(x_n)-f(a)|<\epsilon$, as $|x_n-a|<\delta$.

$\Rightarrow \{f(x_n)\}\longrightarrow f(a)$. Reverse direction in the next lecture...