MATH401: Lecture 11 (09/23/2025)

Midterm exam: Oct 7
Take-home exam; Sections 1.1–1.6, 2.1, 2.2

Today: * Mean value theorem
* Metric spaces

Mean Value Theorem (MVT) on $\mathbb{R}$

For the final theorem (4^{th} one, after IVT, BW, EVT), we assume the function is much "nicer", i.e., it's differentiable, to be able to present a stronger result on its structure. We recall the definition of derivative first.

Recall: **Derivative** of function f at $x=a$ is

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

f is **differentiable** at $x=a$ if the above limit exists.

The mean value theorem says for a differentiable and continuous function f on $[a,b]$, there exists a point inside the interval where the instantaneous slope of the function is equal to the "mean" slope of f over the interval. We need two results to be used as building blocks first.

Lemma 2.3.5 Let $f:[a,b]\to\mathbb{R}$ have a maximum or minimum at an inner point $c\in(a,b)$ where the function is differentiable. Then $f'(c)=0$.

Proof We show $f'(c)>0$ or $f'(c)<0$ is not possible.

Assume $f'(c)>0$. → A similar argument works for $f'(c)<0$.

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \quad \text{by definition.}$$

→ as $f'(c)>0$

$$\Rightarrow \frac{f(x)-f(c)}{x-c}>0 \quad \text{for all } x \text{ sufficiently close to } c.$$

$$\Rightarrow \quad x>c \Rightarrow f(x)>f(c), \text{ and}$$
$$x<c \Rightarrow f(x)<f(c)$$

→ If $x=c$ is a maximum, then $f(x)\le f(c)$ for $\forall x$.

→ The result follows by the contrapositive argument now.

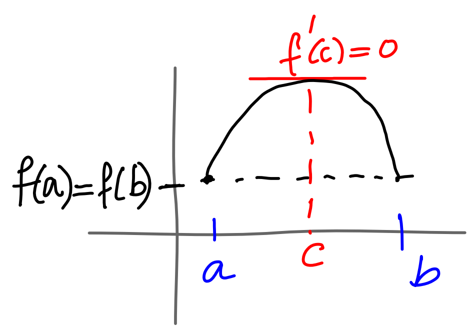
$\Rightarrow x=c$ is neither a maximum nor minimum. □

Lemma 2.3.6 (Rolle's Theorem) Let $f:[a,b]\to\mathbb{R}$ be continuous at all $x\in[a,b]$ and is differentiable at all inner points $x\in(a,b)$. If $f(a)=f(b)$, then there exists a point $c\in(a,b)$ such that $f'(c)=0$.

Proof EVT (Theorem 2.3.4) $\Rightarrow$

f has a maximum and minimum in $[a,b]$. Since $f(a)=f(b)$, at least one of these optima must be at an inner point c.

So Lemma 2.3.5 $\Rightarrow f'(c)=0$. □



Trivial case: $f(x)=f(a)\ \forall x\in[a,b]$

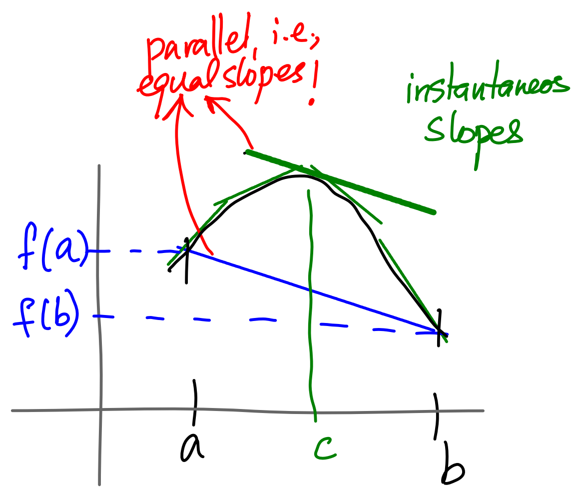
↳ straight line!

Theorem 2.3.7 (The Mean Value Theorem (MVT))

Let $f : [a,b] \to \mathbb{R}$ be continuous in all $[a,b]$ and differentiable at all inner points $x \in (a,b)$. Then there exists $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

The mean (or average) slope of $f(x)$ over $[a,b]$ is $f(b)-f(a)/(b-a)$. This theorem says there is a point $c \in (a,b)$ where the instantaneous slope, i.e., slope of tangent, is equal to the mean slope!



Proof

Let $g(x) = f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x-a)$.

How did we come up with this function ?!! See next page...

$g(a) = f(a)$, and

$g(b) = f(b) - \left(\frac{f(b)-f(a)}{\cancel{b-a}}\right)(\cancel{b-a}) = f(a)$.

$b > a$ by assumption, and hence $b-a \neq 0$.

We can show that $g(x)$ is indeed continuous in $[a,b]$ and differentiable at all $x \in (a,b)$. $\rightarrow$ $g(x) = f(x) + m(x-a)$ for constant m; $f(x)$ is continuous and differentiable, and so is $(x-a)$; their sum is so as well.

So, Rolle's theorem (Lemma 2.3.6) $\Rightarrow \exists\, c \in (a,b)$ s.t. $g'(c) = 0$.

$$\Rightarrow g'(x) = f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right) = 0 \text{ at } x = c.$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}. \qquad \square$$

Now, how did we come up with the $g(x)$ function ?!

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \overset{\text{want to show!}}{=} \frac{f(b)-f(a)}{b-a}$$

$$f'(x)\Big|_{x=c} =$$

$$\Longrightarrow \underbrace{f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)}\Bigg|_{x=c} = 0$$

Looks like $g'(x)=0$ for some function $g(x)$.

We want to find $g(x)$ such that $g(a)=g(b)=0$, and then we could use Rolle's theorem!

So we take antiderivative of $f'(x) - \dfrac{f(b)-f(a)}{b-a}$ to get

$$f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)x + C. \longrightarrow \text{constant}$$

With $g(x) = f(x) - \left(\dfrac{f(b)-f(a)}{b-a}\right)x + C$, we choose c such that $g(a)=g(b)$! Note that

$$\underbrace{f(b) - \left(\frac{f(b)-f(a)}{b-a}\right)b + \left(\frac{f(b)-f(a)}{b-a}\right)a}_{g(b)} = f(a) = \underbrace{f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)a + \left(\frac{f(b)-f(a)}{b-a}\right)a}_{g(a)}.$$

Chapter 3 **Metric Spaces**

We have showed several results on sequences and functions in $\mathbb{R}$ and $\mathbb{R}^m$. But many of these results could be shown for far more general spaces which have many of the nice properties of $\mathbb{R}$ (or $\mathbb{R}^m$). We define metric spaces with this goal in mind.

3.1 Definitions

Def A **metric space** (X, d) consists of a set $X \neq \phi$ (nonempty), and a function $d : X \times X \longrightarrow [0, \infty)$ such that

(i) (positivity) $d(x,y) \geq 0 \ \forall \ x, y \in X$, and
$d(x,y) = 0$ iff $x = y$;

(ii) (symmetry) $d(x,y) = d(y,x) \ \forall \ x,y \in X$; and

(iii) (triangle inequality) $d(x,y) \leq d(x,z) + d(z,y) \ \forall \ x,y,z \in X$.

hold. A function d satisfying (i)–(iii) on X is a **metric** on X.

We sometimes write just X, when the metric d is evident. At the same time, note that a space X could have multiple metrics defined on it. The first example we consider studies a metric on $\mathbb{R}^2$ that is different from the usual Euclidean metric.

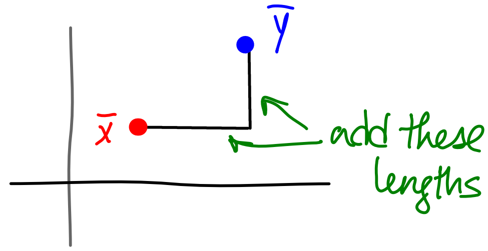
Examples

LSIRA eg. 3   <u>Manhattan or taxicab metric (in $\mathbb{R}^2$).</u> X

For $\bar{x}, \bar{y} \in \mathbb{R}^2$, let $\qquad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ \bar{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$$d(\bar{x}, \bar{y}) = |y_1 - x_1| + |y_2 - x_2|.$$

Check that this is a metric space.



The intuition is that if a taxi is to go from point X to point y in downtown Manhattan with perpendicular streets, it will have to go East/West (horizontal) and then North/South (vertical). We add these two straight line distances to get the taxicab distance between x and Y.

(i) $d(\bar{x}, \bar{y}) \geq 0$ holds, as $|x_1 - y_1| \geq 0$ and $|x_2 - y_2| \geq 0$.
The only way we get $d(\bar{x}, \bar{y}) = 0$ is when both absolute differences are zero, i.e., when $x_1 = y_1$ and $x_2 = y_2$, i.e., when $\bar{x} = \bar{y}$.

(ii) $d(\bar{x}, \bar{y}) = d(\bar{y}, \bar{x})$ follows from absolute differences being symmetric, i.e., $|x_i - y_i| = |y_i - x_i|$ for $i = 1, 2$.

(iii) Triangle inequality:

$$\begin{aligned} d(\bar{x}, \bar{y}) &= |y_1 - x_1| + |y_2 - x_2| \\ &= |y_1 - z_1 + z_1 - x_1| + |y_2 - z_2 + z_2 - x_2| \\ &\leq |y_1 - z_1| + |z_1 - x_1| + |y_2 - z_2| + |z_2 - x_2| \\ &= d(x, z) + d(z, y). \end{aligned}$$

standard triangle inequality in $\mathbb{R}$

Hence $d(\bar{x}, \bar{y}) \leq d(\bar{x}, \bar{z}) + d(\bar{z}, \bar{y}) \ \forall \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^2$. $\square$