MATH 401: Lecture 13 (09/30/2025)

Today: * convergence and continuity in metric spaces.

Convergence and Continuity (LSIRA 3.2)

We can naturally extend the concepts of convergence, functions, and their continuity from $\mathbb{R}$ or $\mathbb{R}^m$ to metric spaces. The only difference is that the distances bounded by ϵ and δ are now measured using the metrics in the metric spaces.

Def 3.2.1 Let (X,d) be a metric space. A sequence $\{x_n\}$ in X converges to $a \in X$ if $\forall \epsilon > 0$ (no matter how small), $\exists N \in \mathbb{N}$ such that $d(x_n,a) < \epsilon \; \forall n \geq N$. We write $\lim_{n\to\infty} x_n = a$, $\{x_n\} \to a$, or $x_n \to a$.

Notice the correspondence to the definition(s) of convergence we have seen previously in $\mathbb{R}$ or $\mathbb{R}^m$. There, $d(x_n,a)$ was replaced by $|x_n - a|$ (in $\mathbb{R}$) or $\|\bar{x}_n - \bar{a}\|$ in $\mathbb{R}^m$.

Def A sequence $\{x_n\}$ in the metric space (X,d) converges to $a \in X$ iff $\lim_{n\to\infty} d(x_n,a) = 0$. (given as Lemma 3.2.2)

We can provide a proof using the standard definition of limit. See LSIRA.

We now talk about functions from one metric space to another, and when they are continuous. We essentially extend the definitions from $\mathbb{R}$ (or $\mathbb{R}^m$) to metric spaces.

Def 3.2.4 Let (X,d_X) and (Y,d_Y) be metric spaces. A function $f: X \to Y$ is continuous at $a \in X$ if
$\forall \epsilon > 0 \; \exists \delta > 0$ such that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x,a) < \delta$.

When talking about $f: \mathbb{R} \to \mathbb{R}$ being continuous, we had both these distances measured as simply $|f(x) - f(a)|$ and $|x-a|$. We are just generalizing those distances to using the corresponding metrics in the spaces here.

LSIRA gives an equivalent definition of continuity in terms of convergence of $\{f(x_n)\}$ to $f(a)$ when $\{x_n\} \to a \in X$. See Proposition 3.2.5.

A Direct Application

Proposition 3.2.6 Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces. Let $f: X \to Y$ and $g: Y \to Z$ be functions, and $h: X \to Z$ be defined as $h(x) = g(f(x))$. If f is continuous at $a \in X$ and g is continuous at $b = f(a) \in Y$, then h is continuous at $a \in X$.

LSIRA presents a proof using Proposition 3.2.5. Here, we give a direct ϵ-δ proof

Problem 2 (pg 51) Prove Proposition 3.2.6 using direct definition of continuity.

Want to show: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_Z(h(x), h(a)) < \epsilon$ whenever $d_X(x, a) < \delta$.

Given f, g are continuous at a and $b = f(a)$, respectively.

$\Rightarrow \forall \epsilon_Y > 0, \exists \delta_X > 0$ s.t. $d_Y(f(x), \underbrace{f(a)}_{=b}) < \epsilon_Y$ whenever $d_X(x, a) < \delta_X$. —— (1)

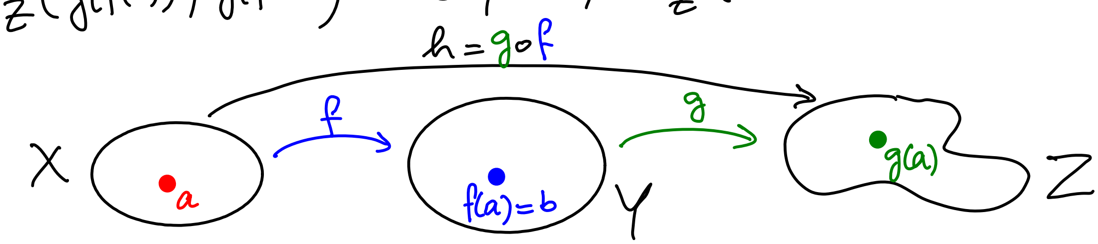
$\forall \epsilon_Z > 0, \exists \delta_Y > 0$ s.t. $d_Z(g(y), g(b)) < \epsilon_Z$ whenever $d_Y(y, b) < \delta_Y$. —— (2)

Let $\epsilon > 0$. (2) $\Longrightarrow$ with $\epsilon_Z = \epsilon$, $\exists \delta_Y > 0$ s.t. $d_Z(g(y), g(b)) < \epsilon$.

(1) $\Longrightarrow$ with $\epsilon_Y = \delta_Y$, $\exists \delta_X$ s.t. $d_Y(f(x), \overset{=b}{f(a)}) < \delta_Y$ whenever $d_X(x, a) < \delta_X$.

$\Rightarrow d_X(x, a) < \delta_X \Rightarrow d_Y(\underset{y}{f(x)}, \underset{b}{f(a)}) < \delta_Y$.

$\Rightarrow d_Z(g(f(x)), g(f(a))) < \epsilon$, i.e., $d_Z(h(x), h(a)) < \epsilon$ as desired. $\square$



A geometric definition of continuity

In general, continuous functions map open sets to open sets. We make this notion more precise here.

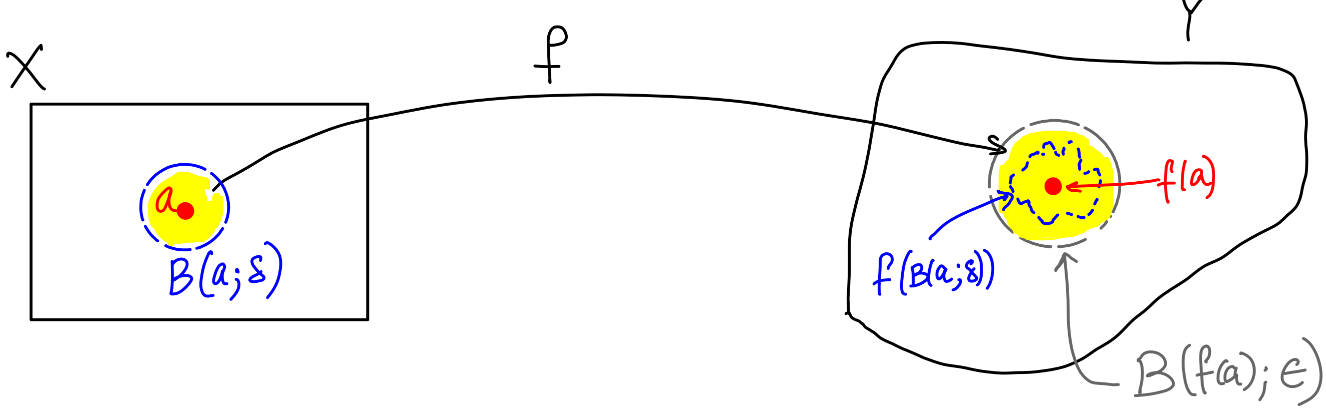
Def (open ball) Let (X, d) be a metric space and $r > 0$, then

$$\bar{B}(a; r) = \{x \in X \mid d(x, a) < r\}$$

$\leq$

is the **open ball** (closed) of radius r centered at $a \in X$.

Some books use $\bar{B}$ to denote the closed ball



Def $f : X \to Y$ is continuous at $a \in X$ if for every open ball $B_Y(f(a); \epsilon)$, $\epsilon > 0$, there is an open ball $B_X(a; \delta)$, $\delta > 0$, such that $f(B(a; \delta)) \subseteq B(f(a); \epsilon)$.

We will use this definition of continuity later on.

Def The function $f : X \to Y$ is **continuous** if it is so at <u>every</u> $x \in X$.

$\longrightarrow$ instead of at just one $a \in X$.

<u>LSIRA Problem 1, pg 51</u> Let (X, d) be the <u>discrete metric space</u>, defined as follows (Example 6, 3.1, pg 46): Let $X \neq \emptyset$, and let

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

We can show $d(\cdot)$ is indeed a metric.

Show that the sequence $\{x_n\} \to a$ <u>iff</u> $\exists N \in \mathbb{N}$ such that $x_n = a \ \forall n \geq N$.

$(\Rightarrow)$ $\exists N \in \mathbb{N}$ s.t. $x_n = a \ \forall n \geq N$.

$$\Rightarrow d(x_n, a) = d(a,a) = 0 \ \forall n \geq N \Rightarrow \{x_n\} \to a.$$

$< \epsilon$ for any $\epsilon > 0$.

$(\Leftarrow)$ $\{x_n\} \to a \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $d(x_n, a) < \epsilon$ whenever $n \geq N$.

Choose $\epsilon = \frac{1}{2}$, and let N_ϵ be its corresponding N.

any number < 1 will do here!

$$\Rightarrow d(x_n, a) < \frac{1}{2} \ \forall n \geq N_\epsilon.$$

But d is the discrete metric, so $d(x_n, a) < \frac{1}{2} \Rightarrow d(x_n, a) = 0$!

But $d(x_n, a) = 0 \Rightarrow x_n = a \ \forall n \geq N_\epsilon$.

$\square$

<u>Problem 5 pg 52</u> Let (X, d) be a metric space. Choose $a \in X$. Show $f: X \to \mathbb{R}$ where $f(x) = d(x,a)$ is a continuous function.

<u>Need to show</u> $f(x)$ is continuous at all points in X.

Let $b \in X$; need to show $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(b)| < \epsilon$ whenever $d(x, b) < \delta$.

Since b is any point in X, f is continuous.

But $|f(x) - f(b)| = |d(x,a) - d(b,a)| \leq d(x, b)$

We know we will have $d(x,b) < \delta$

by <u>inverse triangle inequality</u> (LSIRA Proposition 3.1.4).

By <u>triangle inequality</u> $d(x,a) \leq d(x,b) + d(b,a)$

$$\Rightarrow d(x,b) \geq d(x,a) - d(b,a) \qquad (1)$$

Also, $d(a,b) \leq d(a,x) + d(x,b)$

$$\Rightarrow d(x,b) \geq d(a,b) - d(a,x)$$

$$= d(b,a) - d(x,a) \qquad (2)$$

by symmetry

(1) and (2) $\Rightarrow$ $d(x,b) \geq |d(x,a) - d(b,a)|$.

Hence by choosing $\delta = \epsilon$, we have

$|f(x) - f(b)| < \epsilon$ whenever $d(x,b) < \delta$.

$\Rightarrow f(x)$ is continuous at $b \in X$.

But b is an arbitrary point in X.

$\Rightarrow f(x)$ is continuous. $\square$