MATH401: Lecture 14 (10/02/2025)

Today: * open and closed sets
* review for midterm exam

But first Inverse triangle inequality (LSIRA Proposition 3.1.4)

$$|d(x,a) - d(b,a)| \leq d(x,b) \equiv d(x,b) \geq |d(x,a) - d(b,a)|,$$ i.e., show $d(x,b) \geq d(x,a) - d(b,a)$ and $d(x,b) \geq d(b,a) - d(x,a)$

Proof

By triangle inequality,

$$d(x,a) \leq d(x,b) + d(b,a)$$

$$\Rightarrow \quad d(x,b) \geq d(x,a) - d(b,a) \quad \text{——— (1)}$$

Also, $$d(b,a) \leq d(b,x) + d(x,a)$$

$$\Rightarrow d(b,x) \geq d(b,a) - d(x,a) \quad \text{——— (2)}$$

$= d(x,b)$ by symmetry

(1) & (2) $\Rightarrow \quad d(x,b) \geq |d(x,a) - d(b,a)|.$ □

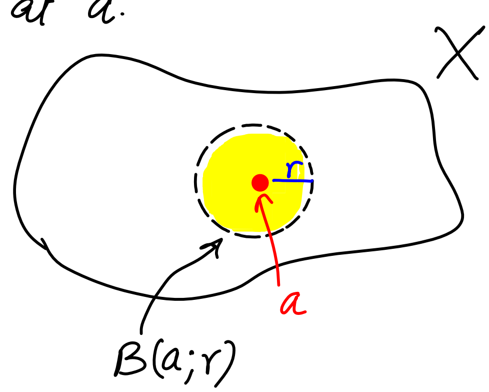
3.3 Open and Closed Sets (in metric spaces)

Recall Ball (open by default): For $a \in (X,d)$, $r > 0$

$$B(a;r) = \{x \in X : d(x,a) < r\}$$ is the open ball of radius r centered at a. Also,

$$\overline{B}(a;r) = \{x \in X : d(x,a) \leq r\}$$ is the closed ball of radius r centered at a.

We draw open balls with dashed border curves, and closed balls with solid boundary/border curves.



Points and Sets

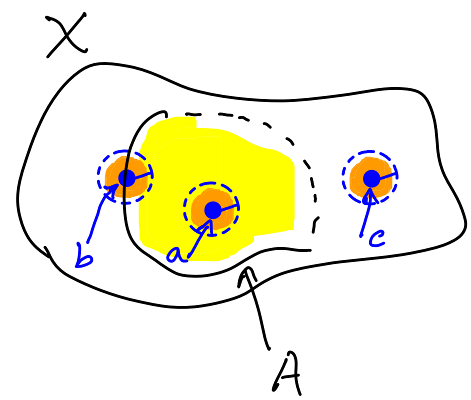
Def Given $x \in X$ and $A \subseteq X$, there are three possibilities.

(i) $\exists\ B(x;r) \subset A$ for $r > 0$; the r-ball at x is contained fully in A

x is an **interior point** of A. e.g., a.

(ii) $\exists\ B(x;r) \subset A^c$ $(= X \setminus A)$

x is an **exterior point** of A, e.g., c.

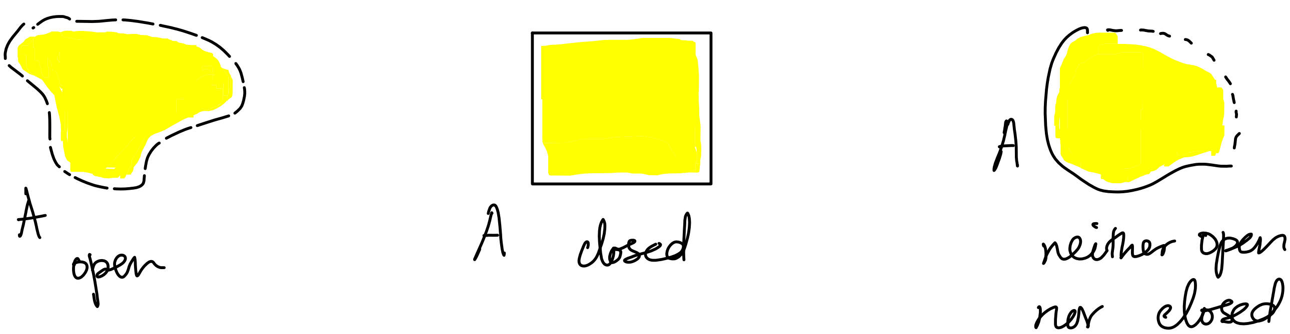


(iii) All balls $B(x;r)$, $r > 0$, intersect both A and A^c.

x is a **boundary point** of A, e.g. b.

The set of all boundary points of A is denoted ∂A, called "boundary of A".

Def A subset A of a metric space (X,d) is **open** if it does not contain any of its boundary points, and it is **closed** if it contains all its boundary points.



ϕ, X are both open and closed, as they do not have any boundary points.

Proposition 3.3.3 A set $\overbrace{A \subset (X,d)}^{\text{set } A \text{ in a metric space}}$ is open iff it consist of only interior points, i.e., $\forall a \in A,\ \exists r > 0$ s.t. $B(a;r) \subset A$.

Proposition 3.3.4 A set $A \subset (X,d)$ is open iff A^c is closed.

A^c ↳ complement; taken w.r.t. X

Proof ($\Rightarrow$) ($\Leftarrow$)

A is open

$\Rightarrow \Leftarrow$ $A \not\ni$ boundary points of A

$\Rightarrow \Leftarrow$ All boundary points of A are in A^c.

$\Rightarrow \Leftarrow$ A^c is closed.

note that boundary points of A are also boundary points of A^c, as every ball centered at these points intersect both A and A^c.

Can present the statements in reverse order for proof in the other direction ($\Leftarrow$)

Given any set A, we can study an associated open set and an associated closed set.

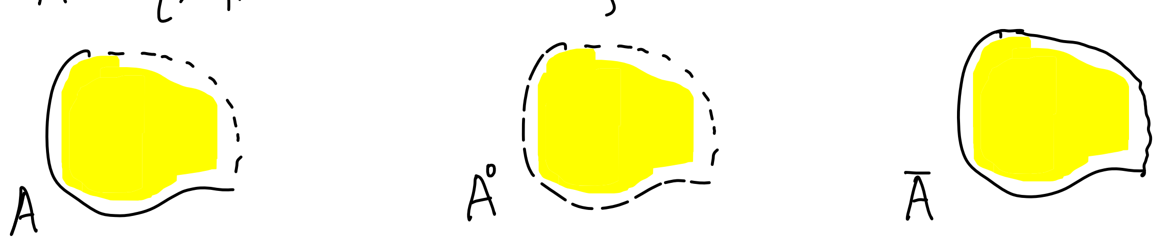
Def The **interior** of $A \subset (X,d)$ is

$$A^\circ = \{x \mid x \text{ is an interior point of } A\},$$

and the **closure** of A is

$$\bar{A} = \{x \mid x \in A \text{ or } x \text{ is a boundary point of } A\}, \text{ or}$$

$$\bar{A} = \{x \mid x \in A \text{ or } x \in \partial A\}.$$



Proposition For any set $A \subseteq (X,d)$, we have $A^\circ \subseteq A \subseteq \bar{A}$.

Think about how you can prove this result.

Proposition 3.3.5 (Problem 4a, Pg 58) A° is open, $\bar{A}$ is closed.

A° is open: A° is the set of interior points of A.

$\Rightarrow \forall x \in A^\circ, \ \exists B(x;r) \subset A, \ r>0.$

$\Rightarrow B(x,r) \cap A^c = \phi.$

$\Rightarrow x$ cannot be a boundary point of A.

Note that $\partial(A^\circ) = \partial A$, as the open balls that intersect A must also intersect A°, by definition.

$\Rightarrow A^\circ$ cannot contain any of its boundary points $\Rightarrow A^\circ$ is open.

Also follows directly from Proposition 3.3.3.

To prove $\bar{A}$ is closed, we prove $\bar{A}^c$ is open. By definition,

$\bar{A}^c = \{x \in X \mid x \notin A \text{ and } x \notin \partial A\}.$ follows from the definition of $\bar{A} = \{x \mid x \in A \text{ or } x \in \partial A\}$.

Let $x \in \bar{A}^c$. $\Rightarrow \exists r>0$ s.t. $B(x;r) \cap A = \phi$. But we want $B(x;r) \subset \bar{A}^c$.

Suppose $y \in B(x;r)$ be s.t. $y \in \partial A. \Rightarrow \exists \epsilon>0$ s.t. $B(y;\epsilon) \cap A \neq \phi$.

definition of boundary point

But $B(y;\epsilon) \subset B(x;r) \Rightarrow B(x;r) \cap A \neq 0$, a contradiction.

$\Rightarrow \forall y \in B(x;r), y \notin A, y \notin \partial A \Rightarrow B(x;r) \subset \bar{A}^c.$

$\Rightarrow x$ is an interior point of $\bar{A}^c$.

$\Rightarrow \bar{A}^c$ is open (by Proposition 3.3.3). $\Rightarrow \bar{A}$ is closed.

$\square$

Quick Review for Midterm

Recall: * injective & surjective functions...

↳ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

* relations and equivalence relations.

↳ reflexive, symmetric, transitive

* countability

↳ may not be necessary to work with a decimal representation to construct a proof for uncountability in all cases.

Check problem from Hw3!

* Convergence

$\{x_n\} \to a$: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $|x_n - a| < \epsilon \ \forall n \geq N$.

* Continuity $f(x)$ is continuous at $x=a$:

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(a)| < \epsilon$ whenever $|x-a| < \delta$.

Recall: fg is continuous when f and g are so.

Want to show: $|f(x)g(x) - f(a)g(a)| < \epsilon$

* Choose ϵ_f, ϵ_g, etc., independent of x & $f(x), g(x)$.

Consider $(|g(a)| + \cancel{\epsilon_g}) \epsilon_f + |f(a)| \epsilon_g$

If one uses ϵ_g here as well, things could be trickier!

e.g., when $g(a) = 0$, $f(a) \neq 0$, we get

$$\underline{\epsilon_g(\epsilon_f + |f(a)|)} \longrightarrow \epsilon$$

↳ harder to choose ϵ_g, ϵ_f to get ϵ!