MATH401: Lecture 16 (10/09/2025)

Today:
* open/closed sets
* continuity using open sets
* completeness in metric spaces

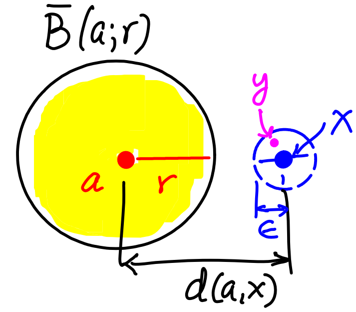
Recall: open and closed sets, interior, boundary, closure of A...

Lemma 3.3.6 $B(a;r)$ is open, and $\bar{B}(a;r)$ is closed. → see LSIRA

We show $\bar{B}(a;r)^c$ is open.

let $x \notin \bar{B}(a;r) \Rightarrow d(a,x) > r$ —— (1)

let $\epsilon = \dfrac{d(a,x)-r}{2}$. —— (2)



Consider $y \in B(x;\epsilon) \Rightarrow d(x,y) < \epsilon$.

$d(a,x) \leq d(a,y) + d(x,y)$ (triangle inequality)

$$\begin{aligned}
\Rightarrow d(a,y) &\geq d(a,x) - d(x,y) \\
&> d(a,x) - \epsilon \\
&= d(a,x) - \left(\frac{d(a,x)-r}{2}\right) \quad \text{by (2)} \\
&= \frac{d(a,x)+r}{2} \\
&> \frac{r+r}{2} = r \quad \text{by (1).}
\end{aligned}$$

$\Rightarrow y \notin \bar{B}(a;r)$; this result holds for any $y \in B(x;\epsilon)$.

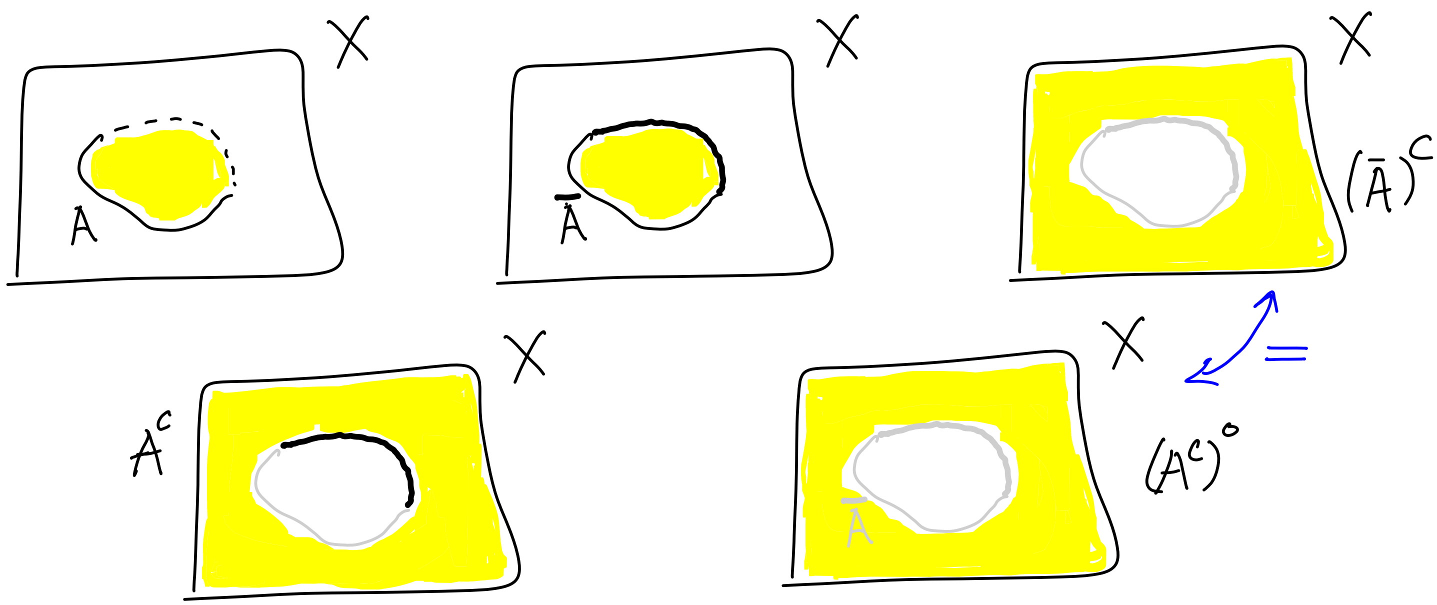
$\Rightarrow B(x;\epsilon) \subseteq \bar{B}(a;r)^c \Rightarrow \bar{B}(a;r)^c$ is open.

$\Rightarrow \bar{B}(a;r)$ is closed. $\square$

We do one more problem before talking about continuity as defined using open sets in metric spaces.

Proposition $(\bar{A})^c = (A^c)^\circ$, where A is a subset of a metric space X.

Here are some illustrations.



($\subseteq$) Let $x \in (\bar{A})^c = X \setminus \bar{A}$

$\Rightarrow x \notin A,\ x \notin \partial A$ $\longrightarrow x \in A^c$

$\Rightarrow \exists\, r > 0$ s.t. $B(x;r) \cap A = \phi$

$\Rightarrow B(x;r) \subset A^c \Rightarrow x \in (A^c)^\circ$.

($\supseteq$) Let $x \in (A^c)^\circ$ $\longrightarrow x \in A^c$

$\Rightarrow \exists\, r > 0$ s.t. $B(x;r) \subset A^c$

$\Rightarrow B(x;r) \cap A = \phi$.

$\Rightarrow x \notin \partial A$, and $x \notin A$

$\Rightarrow x \in (\bar{A})^c$.

$\square$

Proposition 3.3.7 Let $F \subset (X,d)$. The following are equivalent:

(i) F is closed.

(ii) $\forall \{x_n\}$ convergent in F with $a = \lim_{n\to\infty} x_n$, we have $a \in F$.

Proof in LSIRA. Intuitively, a closed set contains all its limit points.

Continuity

We generalize the notion and definitions of continuity in $\mathbb{R}^m$ to metric spaces.

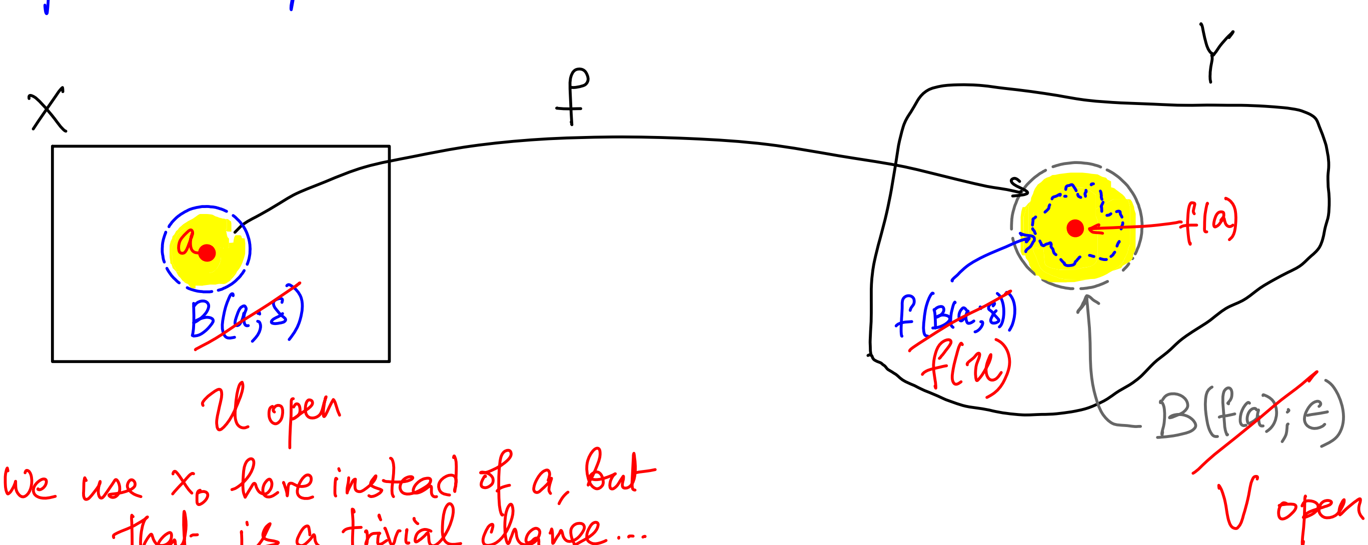
Proposition 3.3.9 Let $f: X \to Y$ be a function, and $x_0 \in X$. (metric spaces)

The following statements are equivalent.

(i) f is continuous at x_0.

(ii) $\forall$ open sets $V \ni f(x_0)$ in Y, $\exists$ open set $\mathcal{U} \ni x_0$ in X s.t. $f(\mathcal{U}) \subseteq V$.

Recall the picture from Lecture 13 — we can consider open sets in place of open balls, and the concepts carry through.



We use x_0 here instead of a, but that is a trivial change...

Proof

(i) $\Rightarrow$ (ii)

f is continuous at $x_0 \Longrightarrow$

$\forall \epsilon > 0, \exists \delta > 0$ s.t. $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$.

Let V be an open set in Y with $f(x_0) \in V$.

$\Rightarrow \exists\ \epsilon > 0$ s.t. $B_Y(f(x_0); \epsilon) \subset V$.

Consider $B_X(x_0; \delta)$; by definition of continuity,

$f(B_X(x_0; \delta)) \subseteq B_Y(f(x_0); \epsilon) \subset V$.

$\Rightarrow \mathcal{U} = B_X(x_0; \delta)$ works for (ii).

(ii) $\Rightarrow$ (i)

Consider $V = B_Y(f(x_0); \epsilon)$ open set containing $f(x_0)$

The result holds for any open set $V \ni f(x_0)$ in Y, so we take $V = B_Y(f(x_0); \epsilon)$

$\exists\ \mathcal{U}$ open, $\mathcal{U} \ni x_0$, s.t. $f(\mathcal{U}) \subseteq V$.

$\mathcal{U}$ open $\Rightarrow \exists\ \delta > 0$ s.t. $B_X(x_0; \delta) \subset \mathcal{U}$.

Take x s.t. $d_X(x, x_0) < \delta \Rightarrow x \in B_X(x_0; \delta) \subseteq \mathcal{U}$

and hence $f(x) \in V = B_Y(f(x_0); \epsilon)$

$\Rightarrow d_Y(f(x), f(x_0)) < \epsilon$.

$\Rightarrow f$ is continuous at x_0, i.e., (i) holds.

$\square$

Continuous functions also map closed sets to closed sets, and this fact is formalized in Proposition 3.3.11.

metric spaces

Proposition 3.3.~~9~~11 Let $f: X \to Y$ be a function, and $x_0 \in X$.

(i) f is continuous at x_0.

(ii) $\forall$ ~~open~~ closed sets $V \ni f(x_0)$ in Y, $\exists$ ~~open~~ closed set $\mathcal{U} \ni x_0$ in X s.t. $f(\mathcal{U}) \subseteq V$.

See LSIRA for proof.

In words, we can replace "open sets" in Prop 3.3.9 with "closed sets" to get Prop 3.3.11.

The book LSIRA specifies definitions of continuity in terms of neighborhoods of x_0 in X and $f(x_0)$ in Y. A neighborhood of x_0 is just an open set containing x_0. But many books define neighborhoods to be either open or closed, but contains an open set that contains x_0.

To avoid any confusion, we refer to open sets containing x_0 (or $f(x_0)$) directly, rather than talk about neighborhoods.

Completeness (LSIRA 3.4)

Recall $\mathbb{R}$ is complete (Section 2.2)
lim sup, lim inf, Cauchy, ...

We generalize the notion of completeness to metric spaces.
It is easier to try and generalize the notion of Cauchy sequences to metric spaces first.

Def 3.4.1 A sequence $\{x_n\}$ in $\underbrace{(X, d)}_{\text{metric space}}$ is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

Proposition 3.4.2 Every convergent sequence in (X, d) is Cauchy.

let $\{x_n\} \to a$ in (X, d) $\Rightarrow \exists N \in \mathbb{N}$ s.t.
$d(x_n, a) < \frac{\epsilon}{2}$ for any $\epsilon > 0$.
We directly start with $\frac{\epsilon}{2}$ here, instead of ϵ

$$\Rightarrow d(x_n, x_m) \leq d(x_n, a) + d(x_m, a) \quad \text{by } \triangle\text{le ineq.}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{whenever } n, m \geq N.$$

$\Rightarrow \{x_n\}$ is Cauchy. $\square$