MATH 401: Lecture 17 (10/14/2025)

Today: * complete metric spaces
* Banach's Fixed Point Theorem (BFPT)

Recall: Every convergent sequence in (X,d) is Cauchy.

But the converse does not always hold.

Example 1

We saw in LSIRA Section 2.2 that $\mathbb{Q}$ is not complete.
With $X = \mathbb{Q}$, and $d(x,y) = |x-y|$, (X,d) is a metric space. → can show all properties of metric spaces hold.

Consider $\{x_n\} = \{1.0, 1.4, 1.41, 1.412, \dots,\} \longrightarrow \sqrt{2} \notin \mathbb{Q}$.
closer and closer approximations of $\sqrt{2}$

Each $x_n \in \mathbb{Q}$, and $\{x_n\}$ is Cauchy. (Why?)

Any pair of elements x_n and x_m are identical upto the $(d-1)^{st}$ decimal digit whenever $n,m \geqslant d$; so $|x_n - x_m| < \frac{1}{10^{(d-1)}}$.

Example 2 $\{\frac{1}{n}\}$, $n \geqslant 2$ is Cauchy in $X = (0,1)$ with $d(x,y) = |x-y|$.

$$|x_n - x_k| = \left|\frac{1}{n} - \frac{1}{k}\right| < \frac{1}{N} \quad \text{whenever } n,k \geqslant N. \text{ So, } N = \left\lceil \frac{1}{\epsilon} \right\rceil$$

will do (for proof that $\{x_n\}$ is Cauchy).

But $\{\frac{1}{n}\} \longrightarrow 0$ as $n \to \infty$, and $0 \notin X = (0,1)$.

So we define a metric space as complete when it includes all limit points.

Def 3.4.3 A metric space (X,d) is called **complete** if all Cauchy sequences in X converge in X.

We are throwing in all limit points to "complete" the space, starting with $X = \mathbb{Q}$, we get $\mathbb{R}$. (Example 1).

Example 2: (continued..) $X = [0,1]$ is complete. Note that $\{x_n\} = \{1 - \frac{1}{n}\} \to 1$ as $n \to \infty$.

In fact, we can formalize this observation — if $A \subset X$ is closed, then it will be complete on its own!

Proposition 3.4.4 Assume (X,d) is a complete metric space. If $A \subset X$, then (A, d_A) is complete iff A is closed.

d_A → restriction of d to A.

($\Leftarrow$) A closed.

Consider a Cauchy sequence $\{a_n\}$ in A.

$\{a_n\}$ is a sequence in X as well, as $A \subset X$.

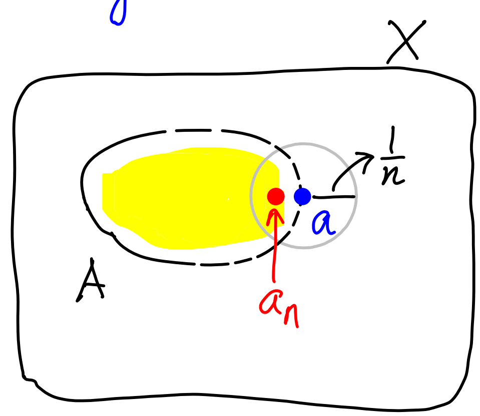
X is complete $\Rightarrow \{a_n\} \to a \in X$.

A is closed $\Rightarrow a \in A$ (by Prop 3.3.7).

$\Rightarrow (A, d_A)$ is complete.

($\Rightarrow$) Let A be not closed. → contrapositive argument

$\Rightarrow \exists\, a \in \partial A,\ a \notin A$ → We identify a Cauchy sequence in A that converges to a.



Pick each $a_n \in B(a; \frac{1}{n})$ such that $a_n \in A$.

$\Rightarrow \{a_n\}$ is Cauchy. (Why?) Also,

$\{a_n\} \to a$ in X, as X is complete, but

$\{a_n\}$ does not converge in A (as $a \notin A$).

$\Rightarrow \exists$ a Cauchy sequence in A that does not converge in A.

$\Rightarrow (A, d_A)$ is not complete.

$\square$

Banach's Fixed Point Theorem (BFPT)

We now present a central result in many areas of mathematics – a fixed point theorem. The theorem will depend crucially on completeness of metric spaces. We first define a fixed point.

Def Let $f: X \to X$ be a function, where (X,d) is a metric space. A point $a \in X$ is a **fixed point** for f if $f(a) = a$.

Motivation In many areas of pure and applied mathematics, we often want to solve $g(\bar{x}) = \bar{0}$. → system of equations

If we can write $g(\bar{x}) = f(\bar{x}) - \bar{x} = \bar{0}$, and study $f(\bar{x}) = \bar{x}$, we are solving a fixed point problem!

For example, $\underbrace{x^5 + 4x^3 - 2}_{g(x)} = 0 \implies x = \underbrace{\left(\frac{2 - x^5}{4}\right)^{1/3}}_{f(x)}$.

We can try to find a sequence $\{\bar{x}_n\}$ where $\bar{x}_{n+1} = f(\bar{x}_n)$ instead of solving $g(x) = 0$ directly. And even if we do not know for sure that $g(x) = 0$ has a (unique) solution, we can take $\bar{x}_n$ as our approximate solution when $n \geq N$ for some large N.

Note that $f(\bar{x})$ may not be unique above – e.g., we could write $x = \underbrace{(2 - 4x^3)^{1/5}}_{f(x)}$ and use a different $f(x)$ to still get $f(x) = x$.

We need one more property of f so as to be able to guarantee the existence of a fixed point.

Def $f: X \to X$ is a **contraction** if $\exists\ 0 < s < 1$ such that $d(f(x), f(y)) \le s\, d(x,y)\ \forall\ x, y \in X$. We say that there is a constant s with $0 < s < 1$

s is the **contraction factor** for f.

Note (i) All contractions are continuous. (Why?)
Can use open ϵ-δ ball definition; choose $\delta = \frac{\epsilon}{s}$.

(ii) $d(f^{\circ n}(x), f^{\circ n}(y)) \le s^n d(x,y)$ where

$$f^{\circ n}(x) = \underbrace{f(f(\cdots f(x)))}_{n \text{ times}} \to n\text{-fold composition of } f$$

We now state and prove Banach's fixed point theorem.

Theorem 3.4.5 (Banach's Fixed Point Theorem)
Let (X,d) be a complete metric space, and $f: X \to X$ be a contraction. Then f has a unique fixed point $a \in X$, and the sequence $\{x_n\}$ converges to a, where $x_0 \in X$ and $x_n = f^{\circ n}(x_0)$, $\forall n \in \mathbb{N}$.

$x_1 = f(x_0)$, $x_2 = f(f(x_0))\ldots$

Proof
We show uniqueness first.
Assume there exist two fixed points $a, b \in X$, $a \ne b$. Then

$$d(a,b) = d(f(a), f(b)) \le s\, d(a,b), \qquad s < 1$$

as a, b are fixed points — as f is a contraction

$$\Rightarrow d(a,b) = 0 \Rightarrow a = b.$$

We prove $\{x_n\}$ is Cauchy. Then $\{x_n\} \to a$, as (X,d) is complete. (some $a \in X$)

Also, $x_{n+1} = f(x_n) \Rightarrow$ as $n \to \infty$, we get
$a = f(a) \Rightarrow a$ is a fixed point.

So we're done if we prove $\{x_n\}$ is Cauchy.

$$d(x_n, x_k) \leq \sum_{i=0}^{k-1} d(x_{n+i}, x_{n+i+1})$$ by $\triangle$le ineq. (see Lecture 12 for a similar result, showed using induction)

$$= \sum_{i=0}^{k-1} d\left(f^{\circ(n+i)}(x_0), f^{\circ(n+i)}(x_1)\right)$$

$$\leq \sum_{i=0}^{k-1} s^{n+i} d(x_0, x_1)$$ Recall, $0<s<1$.

$$= \frac{s^n(1-s^k)}{1-s} d(x_0, x_1)$$ sum of geometric series

$$\leq \frac{s^n}{1-s} \underbrace{d(x_0, x_1)}_{\text{finite}}$$ $s<1$

We can choose $N \in \mathbb{N}$ large enough such that this expression is $< \epsilon$ for any $\epsilon > 0$ whenever $n, k \geq N$ (as $0<s<1$).

$\Rightarrow \{x_n\}$ is Cauchy! $\square$

$$\frac{s^n}{1-s} d(x_0, x_1) < \epsilon$$

$$\Rightarrow s^n < \frac{(1-s)\epsilon}{d(x_0, x_1)}$$

$$-n \log s > -\log\left(\frac{(1-s)\epsilon}{d(x_0, x_1)}\right)$$

$$n \log\left(\frac{1}{s}\right) > \log\left(\frac{d(x_0, x_1)}{(1-s)\epsilon}\right)$$

$$\Rightarrow N \geq \left\lceil \frac{\log\left(d(x_0,x_1)/(1-s)\epsilon\right)}{\log(1/s)} \right\rceil + 1$$ will do.