MATH 401: Lecture 18 (10/16/2025)

Today: * compactness
* relation to closed, bounded, complete sets

Recall BFPT: (X,d) complete, $f: X \to X$ is a contraction:
$\{x_n\} \to a$, where $x_0 \in X$, $x_n = f^{\circ n}(x_0)$ $\forall n \in \mathbb{N}$.

LSIRA Prob 5 (pg 63)

Let $f: [0,1] \to [0,1]$ be differentiable and $\exists$ $0 < s < 1$ s.t. $|f'(x)| < s$ $\forall x \in (0,1)$. Show $\exists$ exactly one $a \in [0,1]$ s.t. $f(a) = a$.

It is evident we want to use Banach's fixed point theorem. In fact, the result is a direct statement of BFPT. So we just need to show f is a contraction, so that BFPT applies.

Let $x, y \in [0,1]$ s.t. $x < y$. Since f is differentiable, we can apply the mean value theorem (MVT). We get that

$$\exists\, c \in (x,y) \text{ s.t. } f'(c) = \frac{f(y) - f(x)}{y - x}.$$

$$\Rightarrow |f(y) - f(x)| = |f'(c)|\,|y - x| < s\,|y - x|$$

(note that $y > x$, so $y - x \neq 0$)

$$\Rightarrow d(f(y), f(x)) < s\, d(y, x) \quad \text{for } s < 1.$$

$$\Rightarrow f \text{ is a contraction.}$$

Hence by BFPT, f has a unique fixed point in $[0,1]$.

We do one more problem on completeness before starting the next topic.

Problem 1, LSIRA Pg 62

$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$

1. Show that the discrete metric space is complete.

Need to show all Cauchy sequences converge.

Let $\{x_n\}$ be Cauchy. Choose $\epsilon = \frac{1}{2}$.

Could use any value < 1 here!

$\Rightarrow \exists\, N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon = \frac{1}{2}$ whenever $m, n \geq N$.

choosing the right ϵ directly

But d is the discrete metric $\Rightarrow d(x_n, x_m) = 0 \Rightarrow x_n = x_m$.

So, with $x_N = a$, we get $x_n = a \;\; \forall n \geq N$.

$\Rightarrow \{x_n\}$ converges to a. $\square$

3.5 Compact Sets

We now talk about another property of sets that make them "nice", and describe its relation to the concepts we have already seen—closedness, boundedness, completeness, etc.

Recall Theorem 2.2.2: Every monotone bounded sequence in $\mathbb{R}$ converges.

We then introduced the concept of Cauchy sequences, and subsequently looked at closed and complete spaces. Recall

metric space

Proposition 3.4.4 $A \subseteq X$, (A, d_A) is complete iff A is closed.

Now, we introduce the concept of compact spaces. We use the idea of subsequences — which we introduced as part of Bolzano-Weierstrass theorem!

Def Let $\{x_n\}$ be a sequence in (X,d) (metric space). For $n_i \in \mathbb{N}$ s.t. $n_1 < n_2 < \cdots < n_i < n_{i+1} < \cdots$ the sequence $\{y_k\} = \{x_{n_k}\}$ is a **subsequence** of $\{x_n\}$.

$$x_1, \underset{1}{x_2}, x_3, \underset{2}{\cdot} \cdots, \underset{k}{x_{n_k}}, \cdots \qquad \{y_k\}: \text{subsequence}$$

Proposition 3.5.1 If $\{x_n\} \to a$, then so do all its subsequences.

Proof (LSIRA Problem 2, Pg 67) Let $\{y_k\} = \{x_{n_k}\}$ be a subsequence.

Note that $k \le n_k$.

So, $k \geq N \Rightarrow n_k \geq k \geq N$.

$$\Rightarrow d(y_k, a) = d(x_{n_k}, a) < \epsilon \quad \forall k \geq N.$$

$$\Rightarrow \{y_k\} \to a. \qquad \square$$

We use the notion of a convergent subsequences to define compact sets.

Def A set $K \subseteq (X,d)$ is a **compact set** if for every sequence in K, there is a subsequence that converges to a point in K.

(X,d) is compact if X is compact in the above sense.

The main point to note is that the limit points are all in K.

Recall: BW theorem: (Proposition 2.3.2) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (also the same result in $\mathbb{R}^m$)

We first explore the relation between compactness and closedness and boundedness. Recall that finite intervals, e.g., $[a,b] \in \mathbb{R}$, are bounded. We define boundedness for metric spaces.

Def $A \subseteq (X,d)$ is **bounded** if there exists $M \in \mathbb{R}$ such that $d(a,b) \le M \ \forall a, b \in A$.

only $0 \le M < \infty$ makes sense

Recall: A closed set A contains all pts in ∂A.

Compactness implies closedness and boundedness.

Proposition 3.5.4 Every compact set K of (X,d) is closed and bounded.

Contrapositive proofs

1. Assume K is not closed.

$\Rightarrow \exists\, a \in \partial K$ s.t. $a \notin K$.

Let $\{x_n\}$ be s.t. $d(x_n, a) < \frac{1}{n}$ and $x_n \in K$.

So $\{x_n\} \to a \notin K$. $\Rightarrow$ All $\{y_k\} = \{x_{n_k}\} \to a \notin K$.

So no subsequence converges to a point in K, i.e., K is not compact.

2. Assume K is not bounded.

$\Rightarrow \nexists\, M$ s.t. $d(a,b) \le M \;\; \forall a, b \in K$.

b is an arbitrary pt. Let $b \in K$. $\forall n \in \mathbb{N}$, $\exists\, x_n \in K$ s.t. $d(x_n, b) > n$.

$\Rightarrow$ For any $\{y_k\} = \{x_{n_k}\}$ $\lim_{k\to\infty} d(y_k, b) = \infty$.

$\Rightarrow \{y_k\}$ cannot converge to a point in K, since

$d(y_k, b) \le d(y_k, a) + d(a,b)$ for any point $a \in K$.

fixed, as a and b are fixed (do not depend on n, k)

by triangle inequality.

$\Rightarrow d(y_k, a) \to \infty$ since $d(y_k, b) \to \infty$ as $k \to \infty$.

$\Rightarrow \{x_n\}$ has no convergent subsequence $\Rightarrow K$ is not compact. $\square$

What about the converse? Holds in $\mathbb{R}^m$!

Corollary 3.5.5 A subset of $\mathbb{R}^m$ is compact iff it is closed and bounded.

Prove only the converse here. $\longrightarrow$ We already saw compact $\Rightarrow$ closed & bounded.

Let A be a closed and bounded set in $\mathbb{R}^m$.

Let $\{\bar{x}_n\}$ be a sequence in A. $\{x_n\}$ is bounded, as A is so.

By BW (Theorem 2.3.3), $\{\bar{x}_n\}$ has a convergent subsequence.

A is closed, so the limit point is in $A \Rightarrow A$ is compact. $\square$

But the converse result does not hold in general for metric spaces.

As an example, consider $(\mathbb{N}, d)$ where d is the discrete metric.

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$$

$\mathbb{N}$ is complete, closed, bounded.

bounded $\rightarrow$ as $d \leq 1$ for discrete metric

closed $\rightarrow$ all candidate limit points are natural numbers

complete $\rightarrow$ all Cauchy sequences converge

But $\{n\}$ does not converge, and nor does any of its subsequences.