MATH401: Lecture 19 (10/21/2025)

Today: * compact v/s complete
* compactness under functions
* total boundedness

Recall Compact $\Rightarrow$ closed and bounded ($\equiv$ in $\mathbb{R}^m$)

What is the relation between compactness and completeness?
e.g., $\mathbb{R}$ is complete, but is not compact!
But all compact sets are complete, as we show below.
In this sense, compactness is the strongest "niceness" property we've seen so far.

Lemma 3.5.6 Let $\{x_n\}$ be a Cauchy sequence in (X,d). If $\exists$ a subsequence $\{x_{n_k}\} \to a$, then $\{x_n\} \to a$ also.
not necessarily complete

Need to show: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $d(x_n, a) < \epsilon \;\; \forall n \geq N$.

Given 1. $\{x_n\}$ is Cauchy.
$\Rightarrow \exists N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \frac{\epsilon}{2} \;\; \forall n,m \geq N$.

2. $\{x_{n_k}\} \to a \Rightarrow \exists K$ s.t. $n_K \geq N$ and
$d(x_{n_K}, a) < \frac{\epsilon}{2}$.

we are directly choosing desired ϵ values here!

$\Rightarrow \forall n, n_K \geq N$

$$d(x_n, a) \leq d(x_n, x_{n_K}) + d(x_{n_K}, a) \quad \text{(By triangle inequality)}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \qquad n_K \geq K \geq N \qquad \square$$

Proposition 3.5.7 Every compact metric space X is complete.

Proof Let $\{x_n\}$ be a Cauchy sequence. Since X is compact, $\{x_n\}$ has a convergent subsequence that converges to $a \in X$ (say). By Lemma 3.5.6, we get that $\{x_n\} \to a$ also. Thus all Cauchy sequences converge, and hence X is complete. $\square$

We next study how compact sets are preserved or not by continuous functions and their inverse images. We get the forward result directly:

Proposition 3.5.9 Let $f : X \to Y$ be continuous.
metric spaces

If $K \subseteq X$ is compact, then $f(K) \subseteq Y$ is compact.

Proof Let $\{y_n\}$ be a sequence in $f(K)$. We want to show it has a convergent subsequence.

We have $y_n \in f(K) \Rightarrow \exists\, x_n \in K$ s.t. $f(x_n) = y_n$.

Consider the sequence $\{x_n\}$ in K. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some $x \in K$. Then $\{y_{n_k}\} = \{f(x_{n_k})\}$ is a subsequence of $\{y_n\}$ that converges to $y = f(x) \in f(K)$ by Proposition 3.2.5 (LSIRA Pg 50). $\square$

This proposition says that for a continuous function $f : X \to Y$ (where X, Y are metric spaces), for all sequences $\{x_n\}$ in X converging to $a \in X$, the sequence $\{f(x_n)\}$ in Y converges to $f(a) \in Y$.

Proposition 3.5.9 says that compact sets get mapped to compact sets by a continuous function. We use this setting to extend the Extreme Value Theorem to arbitrary metric spaces.

Theorem 3.5.10 (Extreme Value Theorem) Let K be a nonempty compact subset of metric space (X,d) and $f: K \to \mathbb{R}$ be continuous. Then f has maximum and minimum points in K, i.e., $\exists\, c, d \in K$ s.t.

$$f(d) \le f(x) \le f(c) \quad \forall\, x \in K.$$

Proof K is compact, f is continuous. So Proposition 3.5.9 gives that $f(K)$ is compact, so it is $\underbrace{\text{closed and bounded}}_{f(K) \subseteq \mathbb{R}}$.

$\Rightarrow \sup f(K), \inf f(K) \in f(K)$ and $\exists\, c, d \in K$ s.t. $f(d) = \inf f(K)$ and $f(c) = \sup f(K)$, i.e., d is a minimum and c is a maximum. $\square$

Compactness may not be preserved under inverse images, as the next problem shows.

LSIRA Problem 8, Pg 68 $f: X \to Y$ is continuous, and let $K \subseteq Y$ is compact. Show that $f^{-1}(K)$ is closed. Find an example where $f^{-1}(K)$ is not compact.

Proof

$K \subseteq Y$ compact $\Rightarrow K$ is closed.

K is closed $\Rightarrow \underline{f^{-1}(K) \text{ is closed.}}$

follows from Proposition 3.3.11, which says continuous functions map closed sets to closed sets.

For the counterexample, consider the following function.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 0$. → could be any constant

$\Rightarrow K = \{0\}$ is closed and bounded, and hence compact (in $\mathbb{R}$).

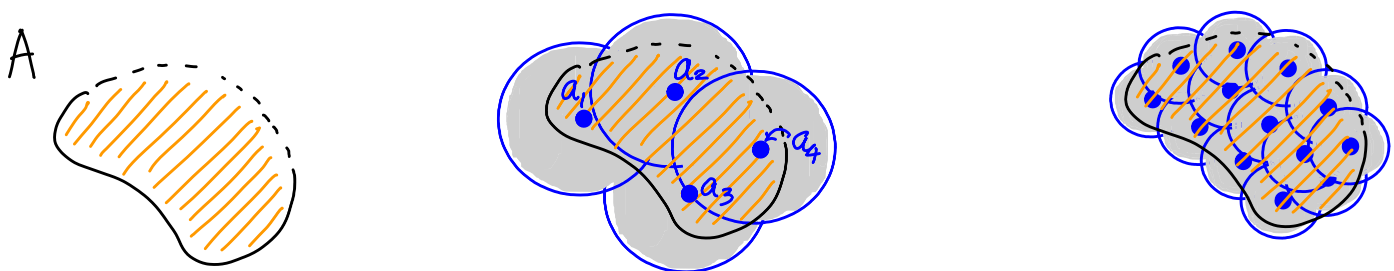
But $f^{-1}(K) = f^{-1}(\{0\}) = \mathbb{R}$ is not bounded, and hence not compact.

Def If f is s.t. $f^{-1}(K)$ is compact whenever K is compact, f is called a **proper** function.

We now introduce a different notion of compactness using open ball covers, which generalizes easily to more general spaces.

Def A set $A \subset (X,d)$ is **totally bounded** if $\forall \epsilon > 0$, there exist finite balls $B(a_i, \epsilon)$, $i = 1, \dots, n$ with $a_i \in A \ \forall i$ that cover A, i.e.,

$$\bigcup_{i=1}^{n} B(a_i, \epsilon) \supseteq A.$$



As ϵ gets smaller, we need to pick more centers $a_i \in A$ to cover A, but it will still be a finite # a_i's.

What is the relationship between compactness and total boundedness? It turns out we get implication in one direction easily.

Proposition 3.5.12 If $A \subseteq (X,d)$ is compact, then it is totally bounded.

<u>Proof</u> We provide a contrapositive proof. We assume A is not totally bounded, and show A is not compact.

We do so by constructing a sequence such that none of its subsequences converge.

Let A be not totally bounded. Then there exists some $\epsilon > 0$ such that no finite collection of ϵ-balls centered in A cover A.

To show A is not compact, we construct a sequence $\{x_n\}$ in A that cannot have a convergent subsequence. We pick

$x_1 \in A$ arbitrarily.

As there is no finite collection covering A.

$B(x_1, \epsilon) \not\supseteq A \Rightarrow$ Can pick $x_2 \in A \setminus B(x_1, \epsilon)$.

$B(x_1, \epsilon) \cup B(x_2, \epsilon) \not\supseteq A \Rightarrow$ Can pick $x_3 \in A \setminus \bigcup_{i=1,2} B(x_i, \epsilon)$.

In general, pick $x_n \in A \setminus \bigcup_{i=1}^{n-1} B(x_i, \epsilon)$.

$\Rightarrow \underline{d(x_n, x_m) \geq \epsilon} \;\; \forall \, n, m \in \mathbb{N}, \; n \neq m.$

Each point x_n is chosen outside of all previous $(n-1)$ ϵ-balls, and hence is $\geq \epsilon$ away from $a_1, \dots, a_{n-1}$.

So $\{x_n\}$ is not <u>Cauchy, and hence not convergent.</u>

Proposition 3.4.2: Every convergent sequence is Cauchy.

But we need to ensure that none of its subsequences converge as well. And we do get that for the same reason!

$\{x_n\}$ cannot have a convergent subsequence.

We can pick any subsequence of $\{x_n\}$ here, say $\{y_k\} = \{x_{n_k}\}$.

$$\Rightarrow d(y_k, y_\ell) = d(x_{n_k}, x_{n_\ell}) \geq \epsilon \;\; \forall \, k, \ell \in \mathbb{N}$$

$\Rightarrow \{y_k\}$ is not Cauchy. $\Rightarrow \{y_k\} = \{x_{n_k}\}$ is not convergent. $\square$