# MATH 401: Lecture 1 (08/19/2025)

This is Introduction to Analysis I

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Today: * syllabus, logistics → see the course web page for details
* proof techniques
  - contrapositive proof
  - proof by contradiction
  - proof by induction

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Book: Lindstrøm: Spaces - An Intro to Real Analysis (LSIRA)

LSIRA 1.1

Logical statements and notation.

If A then B (or $A \Rightarrow B$) "implies"

$A \Rightarrow B$ typically does not mean $B \Rightarrow A$.

e.g., A: $p$ a natural number, is divisible by 6

B: $p$ is divisible by 3.

$A \Rightarrow B$ holds, but $B \not\Rightarrow A$ (B does not imply A),

e.g., $p = 9$.

But if $A \Rightarrow B$ and $B \Rightarrow A$ hold, we say A $\underbrace{\text{if and only if}}_{\text{iff}}$ B, or

$A \Leftrightarrow B$ (or A is equivalent to B).

To prove $A \Leftrightarrow B$, we often prove $A \Rightarrow B$ and $B \Rightarrow A$

$(A \Leftarrow B)$ separately.

We start by reviewing certain standard techniques to construct proofs of mathematical statements.

# 1. Contrapositive Proof

To show $A \Rightarrow B$, equivalently show not $B \Rightarrow$ not $A$ ($\neg B \Rightarrow \neg A$).

$\neg$ "negation" or "not"

"If A happened then B happened"
This statement is equivalent to
"If B did not happen then A did not happen."

LSIRA 1.1 Prob 3. Prove the following Lemma.

**Lemma 1** If $n$ is a natural number such that $n^2$ is divisible by 3, then $n$ is divisible by 3.

This is $A \Rightarrow B$ where $A$: $3 \mid n^2$ ($n^2$ is divisible by 3). (or 3 divides $n^2$)
$B$: $3 \mid n$ ($n$ is divisible by 3).

Let's try to prove $A \Rightarrow B$ directly: $n^2 = 3k$ (as $n^2|3$) $\Rightarrow$ $n = \sqrt{3}\sqrt{k}$ (taking square root on both sides)

Hard to conclude that $n|3$ ☹! Would have to argue $k|3$, which is not obvious!

Let's try proving $\neg B \Rightarrow \neg A$.

$\neg B$: $n$ is not divisible by 3.

$\Rightarrow$ $n = 3p+1$ or $n = 3q+2$, for $p, q \in \mathbb{N}$. (set of natural numbers)

Case 1. $n = 3p+1$

$$\Rightarrow n^2 = (3p+1)^2$$
$$= 9p^2 + 6p + 1$$
$$= 3(3p^2+2p)+1$$
$$= 3k+1 \text{ for } k = 3p^2+2p$$

$\Rightarrow n^2$ is not divisible by 3

Case 2. $n = 3q+2$

$$\Rightarrow n^2 = (3q+2)^2$$
$$= 9q^2 + 12q + 4$$
$$= 9q^2 + 12q + 3 + 1$$
$$= 3\underbrace{(3q^2+4q+1)}_{=k'}+1$$
$$= 3k'+1$$

$\Rightarrow n^2$ is not divisible by 3.

Hence we have proved that if $n$ is not divisible by 3, then $n^2$ is not divisible by 3. Hence, by the contrapositive, we have $n^2|3 \Rightarrow n|3$. $\square$

Should we always try to build a contrapositive proof?
Not necessarily! In cases where $A \Rightarrow B$ could be concluded directly, the contrapositive argument might make life harder!
It is one of the different proof approaches that you should be aware of.

## 2. Proof by Contradiction

Assume opposite of what you want to prove, and end up with a contradiction (or an obviously wrong statement). Hence the original assumption must be wrong, i.e., you have proved the statement.

<u>LSIRA 1.1 Prob 3 (continued)</u> Prove the following Theorem.

<u>Theorem 2</u> $\sqrt{3}$ is irrational.

Assume $\sqrt{3}$ is rational. → the opposite of what you want to prove

$\Rightarrow \left(\sqrt{3} = \frac{p}{q}\right)^2$, $p, q \in \mathbb{N}$ with no common factors.

→ by definition, any positive rational number can be written in the form $p/q$, as specified.

Let's square both sides, and cross multiply.

$\Rightarrow 3q^2 = p^2 \Rightarrow 3 \mid p^2$ ($p^2$ is divisible by 3).

Hence by Lemma 1, $3 \mid p$. Let $p = 3k$. ($k \in \mathbb{N}$). Plug $p=3k$ back in:

$\Rightarrow 3q^2 = (3k)^2 = 9k^2$ (divide both sides by 3)

$\Rightarrow q^2 = 3k^2$, i.e., $3 \mid q^2$ ($q^2$ is divisible by 3).

Again by Lemma 1, $3 \mid q$.

Since we started with the assumption that p and q have no common factors

Thus p and q have a common factor of 3, which is a contradiction.

Hence $\sqrt{3}$ is irrational. $\square$

# 3. Proof by Induction

To show a statement $P(n)$ holds for all $n \in \mathbb{N}$,

1. show $P(1)$ holds;
2. Assume $P(k)$ holds for some $k \in \mathbb{N}$.
3. Show $P(k+1)$ holds under Assumption 2.

"for all"

Example

Show that $P(n) = 3 + 5 + \cdots + 2n+1 = n(n+2) \ \forall n \in \mathbb{N}$.

1. $P(1) = 3 = 1(1+2)$ (so $P(1)$ is true).
2. Assume $P(k) = k(k+2)$ for some $k \in \mathbb{N}$.
3. $P(k+1) = P(k) + 2(k+1)+1 = P(k) + 2k+3$

$$= k(k+2) + 2k+3 \quad \text{by induction assumption.}$$
$$= k(k+2) + k + k+3$$
$$= k(k+3) + k+3$$
$$= (k+1)(k+3) = n(n+2) \text{ for } n = k+1.$$

$\Rightarrow P(n) = n(n+2) \ \forall n \in \mathbb{N}$. □