MATH 401: Lecture 24 (11/06/2025)

Today: * $\frac{d}{dx}$ of $\{f_n(x)\}$
* spaces of bounded functions

Recall: $\left\{\frac{\sin nx}{n}\right\} \xrightarrow{\text{uniformly}} 0$ but $\{\cos nx\}$ does not converge.

How do we show the first result? Can use Proposition 4.2.3!
$\sup\{d_Y(f_n(x), f(x)) \mid x \in X\} \to 0$ as $n \to \infty$. Here, the limit function $f(x) = 0$, $|\sin nx| \leq 1 \ \forall x \in X$, and $\left|\frac{\sin nx}{n}\right| \leq \frac{1}{n} \to 0$ as $n \to \infty$.

Problem 5, LSIRA pg 91-92

a) Show that $\displaystyle \frac{1}{1-e^{-x}} = \sum_{n=0}^{\infty} e^{-nx}$ for $x > 0$.

b) Explain we can differentiate term-by-term to get
$$\frac{e^{-x}}{(1-e^{-x})^2} = \sum_{n=1}^{\infty} n e^{-nx} \quad \forall x > 0.$$

c) Does the series $\displaystyle \sum_{n=1}^{\infty} n e^{-nx}$ converge uniformly on $[0,\infty)$?

a) Recall: $\displaystyle \sum_{n=0}^{\infty} k^n = \frac{1}{1-k}$ when $|k| < 1$.

and $\displaystyle \sum_{n=0}^{N} k^n = \frac{k^{N+1}-1}{k-1}$.

$x > 0 \Rightarrow e^{-x} = \frac{1}{e^x} < 1 \Longrightarrow$
$$\sum_{n=0}^{\infty} e^{-nx} = \sum_{n=0}^{\infty} (e^{-x})^n = \frac{1}{1-e^{-x}}.$$

(b) We consider partial sums up to N terms, and then let $N \to \infty$.

$$\text{Let } S_N(x) = \sum_{n=0}^{N} e^{-nx} = \frac{\overset{\text{finite!}}{e^{-(N+1)x}} - 1}{e^{-x} - 1}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

both the LHS and RHS expressions are finite term functions of x, which we can differentiate.

$$\Rightarrow \quad S_N'(x) = \sum_{n=0}^{N} -ne^{-nx} =$$

$$\frac{-(N+1)\,\underbrace{e^{-(N+1)x}}_{=0 \text{ as } N\to\infty}(e^{-x}-1) - (-e^{-x})(\underbrace{e^{-(N+1)x}}_{=0 \text{ as } N\to\infty} - 1)}{(e^{-x}-1)^2}$$

We take limit as $N \to \infty$ on both sides to get

$$\lim_{N\to\infty} -S_N'(x) = \lim_{N\to\infty}\left(\sum_{n=0}^{N} ne^{-nx}\right) = \sum_{n=0}^{\infty} ne^{-nx}$$

$$= -\left(\frac{\cancel{(N+1)(0)(e^{-x}-1)} + e^{-x}(0-1)}{(e^{-x}-1)^2}\right) = \frac{e^{-x}}{(e^{-x}-1)^2}.$$

(c) Recall that we consider all $x \in X$ for uniform continuity. In this case, we consider $x \to 0$ in the equation in (b) above:
$\hookrightarrow$ study both sides for small x values

$$\lim_{x \to 0} \left(\frac{e^{-x}}{(1-e^{-x})^2} - \sum_{n=0}^{N} n e^{-nx} \right)$$

$$= \frac{1}{(1-1)^2} - \underbrace{\sum_{n=0}^{N} n}_{\text{finite}} = \infty$$

Hence we do not get uniform convergence here!
$\hookrightarrow$ Recall Proposition 4.2.3 which characterizes uniform convergence in terms of $\sup\{d_Y(f_n(x), f(x)) \mid x \in X\} \to 0$ as $n \to \infty$.

We finish the discussion with result that guarantees when we can differentiate a series term-by-term.

Proposition 4.3.5 Let $\{f_n(x)\}$ be a sequence of differentiable functions on $[a,b]$. Assume $f_n'(x)$ are continuous and converge uniformly to $g(x)$ on $[a,b]$. Assume $\exists\, x_0 \in [a,b]$ such that $\{f_n(x_0)\}$ converges. Then $\{f_n(x)\}$ converges uniformly on $[a,b]$ to a differentiable function $f(x)$ s.t. $f'(x) = g(x)$.

This result can be presented for series (Corollary 4.3.6) – see LSIRA.

4.5 Spaces of Bounded Functions

We now study metric spaces of functions, i.e., metric spaces where the elements themselves are functions. We explore how to generalize concepts of completeness, compactness, etc. this space.

The motivation comes from applications to other areas of mathematics. For instance, with a version of Banach's Fixed Point Theorem in this space, we could prove existence of solutions to differential equations!

We need to assume the functions are "nice" so as to be able to define a metric space. We start with boundedness. In the next section, we will add continuity.

Def Let (X,d_X), (Y,d_Y) be metric spaces. A function $f: X \to Y$ is **bounded** if $\{f(x) : x \in X\}$ is a bounded set in Y, i.e., $\exists\, M \in \mathbb{R}$ s.t. $d_Y(f(u),f(v)) \le M \;\; \forall u,v \in X$.

Equivalently, $\forall a \in X$, $\exists$ constant M_a s.t. $d_Y(f(a),f(x)) \le M_a$ $\forall x \in X$. Can take $M = \sup\{M_a \mid a \in X\}$ here.

The next result says that distance between bounded functions is bounded.

Proposition Let $f,g: X \to Y$ be bounded functions. Then there exists a K s.t. $d_Y(f(x),g(x)) \le K \;\; \forall x \in X$.

Proof

Fix $a \in X$, and let M_a, N_a be such that (both f,g are bounded)

$$d_Y(f(x),f(a)) \le M_a \text{ and } d_Y(g(x),g(a)) \le N_a \quad \forall x \in X.$$

$$\Rightarrow d_Y(f(x),g(x)) \underbrace{\le d_Y(f(x),f(a)) + d_Y(f(a),g(a)) + d_Y(g(a),g(x))}_{\text{triangle inequality.}}$$

$$\le M_a + \underbrace{d_Y(f(a),g(a))}_{\text{finite constant}} + N_a = K.$$

$\square$

The previous proposition allows us to define a space of bounded functions.

Def $B(X,Y)=\{f:X\to Y \mid f \text{ is bounded}\}$ is the space of bounded functions.

Let $\rho(f,g)=\sup\{d_Y(f(x),g(x)) \mid x\in X\}$.

Then $\rho(f,g)<\infty \ \forall f,g\in B(X,Y)$.

Proposition 4.5.1 ρ is a metric, and $(B(X,Y),\rho)$ is a metric space.

Finiteness, positivity, and symmetry are straightforward to show.

Triangle inequality: For $f,g,h\in B(X,Y)$,

$$d_Y(f(x),g(x))\le \underline{d_Y(f(x),h(x))+d_Y(h(x),g(x))} \quad \forall x\in X$$

→ triangle inequality in Y.

Taking sup over all $x\in X$ gives

$$\rho(f,g)\le \rho(f,h)+\rho(h,g).$$

Problem 1, LSIRA pg 100 Let $f,g:[0,1]\to\mathbb{R}$ be $f(x)=x$, $g(x)=x^2$. Find $\rho(f,g)$.

$$\rho(f,g)=\sup_{x\in[0,1]}\{|x-x^2|\}.$$

With $h(x)=x-x^2$, we can find max $h(x)$ over $[0,1]$.

$$h'(x)=1-2x=0 \implies x=\tfrac{1}{2}$$

$$h''(x)=-2<0 \implies x=\tfrac{1}{2} \text{ is a local maximum.}$$

$$h(\tfrac{1}{2})=\left|\tfrac{1}{2}-\left(\tfrac{1}{2}\right)^2\right|=\tfrac{1}{4}. \quad \text{Also, } h(0)=0,\ h(1)=0.$$

$$\implies \rho(f,g)=\tfrac{1}{4}.$$

What about convergence in $B(X,Y)$?

Proposition 4.5.2 A sequence $\{f_n\}$ converges to f in $(B(X,Y),\rho)$ iff $\{f_n\}$ converges uniformly to f.

converges → convergence in the ρ-metric

converges → convergence in d_Y metric.

Proof By Proposition 4.2.3, $\{f_n\}$ converges uniformly to f iff

$$\sup\{d_Y(f_n(x), f(x)) \mid x \in X\} \longrightarrow 0$$

$$\Longleftrightarrow \quad \rho(f_n, f) \longrightarrow 0.$$