MATH 401: Lecture 25 (11/13/2025)

Today: * spaces of bounded and continuous functions

Recall: $(B(X,Y), \rho)$; $f \in B(X,Y) \Rightarrow f: X \to Y$ is bounded; $\rho(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}$

We saw (Prop 4.5.2) that $\{f_n\} \xrightarrow[\rho\text{-metric}]{} f \Leftrightarrow \{f_n\} \xrightarrow[d_Y \text{ metric}]{\text{uniformly}} f_n$.

What about Completeness? Yes, if Y is complete!

Theorem 4.5.3 Let (X, d_X), (Y, d_Y) be metric spaces. Assume (Y, d_Y) is complete. Then $(B(X,Y), \rho)$ is also complete.

Proof Let $\{f_n\}$ be a Cauchy sequence in $B(X,Y)$.

Want to show $\{f_n\} \to f \in B(X,Y)$. from definition of $\rho(f_n, f_m)$

Let $x \in X$. Then $d_Y(f_n(x), f_m(x)) \leq \rho(f_n, f_m)$ and $\{f_n\}$ is a Cauchy sequence in $B(X,Y) \Rightarrow \rho(f_n, f_m) < \epsilon$ whenever $n, m \geq N$ for some $N \in \mathbb{N}$.

$\Rightarrow$ $\{f_n(x)\}$ as function values is a Cauchy sequence in Y.

Y is complete $\Rightarrow$ $\{f_n(x)\} \to f(x)$ in Y.

Need to show: $f \in B(X,Y)$ and

$\{f_n\} \to f$ in the ρ-metric.

We already saw that since $\{f_n\}$ is a Cauchy sequence in $(B(X,Y), \rho)$,

$\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\rho(f_n, f_m) < \epsilon$ when $n,m \geq N$, $\forall x \in X$.

$\Rightarrow d_Y(f_n(x), f_m(x)) < \epsilon \quad \forall n,m \geq N, \forall x \in X$.

$\Rightarrow d_Y(f_n(x), f(x)) = \underline{\lim_{m\to\infty}} d_Y(f_n(x), f_m(x)) < \epsilon \quad \forall x \in X$.

keep n fixed, let $m \to \infty$; $m \geq N$ still holds

$\Rightarrow \rho(f_n, f) < \epsilon$

$\Rightarrow f$ is bounded! Why? $\rightarrow$ Problem 5, LSIRA pg 101

arbitrary anchor point

Let $a \in X$. f_n is bounded $\Rightarrow \exists M_a$ such that

$$d_Y(f_n(a), f_n(x)) \leq M_a \quad \forall x \in X.$$

Also, $\rho(f_n, f) < \epsilon \Rightarrow d_Y(f_n(a), f(a)) < \epsilon$ and $d_Y(f_n(x), f(x)) < \epsilon$.

$$\Rightarrow d_Y(f(a), f(x)) \leq d_Y(f(a), f_n(a)) + d_Y(f_n(a), f_n(x)) + d_Y(f_n(x), f(x))$$

$$< \epsilon + M_a + \epsilon = M, \text{ showing } f \text{ is bounded.}$$

$\Rightarrow f \in B(X,Y)$, and as $\rho(f_n, f) < \epsilon$ for any $\epsilon > 0$, we get that $\{f_n\}$ converges to f in the ρ-metric. $\square$

Recall: $B(X,Y) = \{f : X \to Y \mid f \text{ is bounded}\}$.

We now consider continuous functions. First, we assume boundedness too.

Def Let (X,d_X), (Y,d_Y) be metric spaces. We define

$$C_b(X,Y) = \{f : X \to Y \mid f \text{ is continuous and bounded}\}.$$

We get $C_b(X,Y) \subseteq B(X,Y)$, → contains bounded functions that are not continuous as well

and hence

$\rho(f,g) = \sup\{d_Y(f(x),g(x)) \mid x \in X\}$ is a metric also on $C_b(X,Y)$.

→ restriction of $\rho(f,g)$ for $B(X,Y)$ to $C_b(X,Y)$

Continuous functions are "nice" already. Do we need boundedness?

Def $C(X,Y) = \{f : X \to Y \mid f \text{ is continuous}\}$ is the set of all continuous functions from X to Y.

But we could have some trouble defining a metric for $C(X,Y)$!

Problem 1, LSIRA pg 102 Let $X, Y = \mathbb{R}$. Find functions $f, g \in C(X,Y)$ such that $\sup\{d_Y(f(x),g(x)) \mid x \in X\} \to \infty$.

Let $f(x) = 0$, $g(x) = x$. Both are continuous. But

$$d_Y(f(x),g(x)) = |f(x) - g(x)| = |x| \to \infty \text{ as } x \to \infty.$$

Or take $f(x) = x$, $g(x) = x^2$ ...

So we turn back to $C_b(X,Y)$. It has nice structure!

Proposition 4.6.1 $C_b(X,Y)$ is a closed subset of $B(X,Y)$.

Proof If $\{f_n\} \in C_b(X,Y)$ is a sequence that converges to $f \in B(X,Y)$, then we show $f \in C_b(X,Y)$.

i.e., show f is continuous

See Lecture 16

We are using Proposition 3.3.7 here, which says a closed subset of a metric space contains all its limit points.

Lecture 24

By Proposition 4.5.2, we know $\{f_n\}$ converges uniformly to f.

Each f_n is continuous (as it is in $C_b(X,Y)$).

Then Proposition 4.2.4 $\Rightarrow$ f is continuous, i.e., $f \in C_b(X,Y)$. $\square$

Lecture 22

We get completeness for $C_b(X,Y)$ when Y is complete. Recall that Theorem 4.5.3 gave completeness for $(B(X,Y),\rho)$ when Y is complete.

Theorem 4.6.2 Let (X,d_X), (Y,d_Y) be metric spaces. Let Y be complete. Then $C_b((X,Y),\rho)$ is also complete.

Lecture 17

Proposition 3.4.4: A subspace of a complete (X,d) is complete iff it is closed.

Theorem 4.5.3: $B((X,Y),\rho)$ is complete (as Y is).

And Proposition 4.6.1: $C_b(X,Y)$ is a closed subset of $B(X,Y)$.

$\Rightarrow$ $C_b(X,Y)$ is complete. $\square$

Can we get boundedness without explicitly assuming the same?
Yes! When X is compact.

Proposition 4.6.3 Let (X, d_X), (Y, d_Y) be metric spaces, and let X be compact. Then all continuous functions from X to Y are bounded.

Proof Let $f: X \to Y$ be continuous. Pick $a \in X$. Consider $h(x) = d_Y(f(x), f(a))$. We show h is bounded.

X is compact, so if we can show h is continuous, then by Theorem 3.5.10 (EVT), we get that h is bounded.

$$
\begin{aligned}
|h(x) - h(y)| &= |d_Y(f(x), f(a)) - d_Y(f(y), f(a))| \\
&\leq d_Y(f(x), f(y)) \quad \text{by inverse } \triangle \text{ ineq.} \\
&< \epsilon
\end{aligned}
$$

as f is continuous (same δ can be used for any given $\epsilon > 0$). $\square$

Thus, if X is compact and Y is complete, then we have both nice properties: $C(X, Y) = C_b(X, Y)$ and $(C_b(X, Y), \rho)$ is complete. This is the typical setting when we take $X = [a, b] \subset \mathbb{R}$ and $Y = \mathbb{R}$, for instance.

Problem 3, LSIRA pg 102 Let $f \in C_b(\mathbb{R}, \mathbb{R})$ bounded, continuous

and $u \in C([0,1], \mathbb{R})$. continuous

Define $L(u) : [0,1] \to \mathbb{R}$ as

$$L(u)(t) = \int_0^1 \frac{1}{1+t+s} f(u(s))\,ds.$$

$L(u)$ is a function of a function $u(t)$

a) Show $L(u) \in C([0,1], \mathbb{R})$. i.e., it's continuous.

We want to show: $\forall \epsilon > 0, \exists \delta > 0$ s.t. $\circledast < \epsilon$ whenever $|x-a| < \delta$

Fix $a \in [0,1]$

$$\overset{\circledast}{\left|L(u)(x) - L(u)(a)\right|} = \left| \int_0^1 \left(\frac{1}{1+x+s} f(u(s)) - \frac{1}{1+a+s} f(u(s)) \right) ds \right|$$

$$= \left| \int_0^1 f(u(s)) \frac{a-x}{(1+x+s)(1+a+s)} ds \right| \qquad \frac{(1+a+s)-(1+x+s)}{(1+x+s)(1+a+s)}$$

f is bounded $\implies |f(u(s))| \leq M \quad \forall s \in [0,1] \implies$

$$\leq |a-x| M \left| \int_0^1 \frac{ds}{(1+x+s)(1+a+s)} \right|$$

$(1+x+s)(1+a+s) \geq (1+s)^2$ as $x, a \in [0,1] \implies$

$$\leq |x-a| M \left| \int_0^1 \frac{ds}{(1+s)^2} \right|$$

we get an upper bound in this manner, and will pick δ (for $|x-a| < \delta$) using that upper bound

change of variable:

$1+s = u$

$ds = du$

$s : 0 \to 1$

$\equiv u : 1 \to 2$

$$= |x-a|M\left|\int_1^2 \frac{du}{u^2}\right| = |x-a|M\left|\left.\frac{-1}{u}\right|_1^2\right| = |x-a|M\cdot\left(\frac{1}{1}-\frac{1}{2}\right)$$
$$= |x-a|\frac{M}{2}.$$

With $|x-a|<\delta$, pick $\delta = \frac{2\epsilon}{M}$, and we get

$$\left|L(u)(x) - L(u)(a)\right| < \left(\frac{2\epsilon}{M}\right)\left(\frac{M}{2}\right) = \epsilon, \quad \text{whenever } |x-a|<\delta.$$

(b) Application of BFPT.

assigned in last homework!