MATH 401 : Lecture 27 (11/20/2025)

Today: * Compact subsets of $C(X,Y)$

4.8 Compact Sets of Continuous Functions

In $\mathbb{R}^m$, compact $\iff$ closed and bounded, and the latter two conditions are easy to check. We would like to give a similar characterization for compact sets of $C(X,\mathbb{R}^m)$, that is also easy to check. We'll see Compact subsets of $C(X,\mathbb{R}^m) \iff$ closed, bounded, and equicontinuous.

We first introduce another related concept.

Def Let (X,d) be a metric space. A subset $D \subseteq X$ is **dense** iff $\forall x \in X, \forall \delta > 0, \exists y \in D$ s.t. $d(x,y) < \delta$.

For instance, $\mathbb{Q}$ is dense in $\mathbb{R}$.
Any $a \in \mathbb{R}$ ($\mathbb{R}/\mathbb{Q}$ to be nontrivial) can be approximated arbitrarily well using $q \in \mathbb{Q}$.

e.g., $\pi = 3.1416\ldots$ $\pi_1 = 3.1 = \frac{31}{10}$; $\pi_2 = 3.14 = \frac{314}{100}$; $\pi_3 = 3.141 = \frac{3141}{1000}, \ldots$

Recall, $\mathbb{R}$ is uncountable, but $\mathbb{Q}$ is countable, and is dense!
We study this property in general — a set is uncountable, but has a dense subset that is countable.

Def A metric space (X,d) is **separable** if X has a countable and dense subset A.

e.g., $\mathbb{R}$ is separable, as $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.

<u>Problem 1, LSIRA Pg 111</u> Show $\mathbb{R}^n$ is separable for all n.

$n=1$: $\mathbb{Q}$ is countable, and dense in $\mathbb{R}$. ✓

By Proposition 1.6.1, the Cartesian product of a finite number of countable sets is countable.

$$\Longrightarrow \mathbb{Q}^n \text{ is countable.}$$

We argue than $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$ by showing

$\forall \bar{x} \in \mathbb{R}^n$, $\exists\ \bar{q} \in \mathbb{Q}^n$ such that $\|\bar{x}-\bar{q}\| < \delta$ for any given $\delta>0$.

$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\bar{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ here. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can

find q_i such that $|x_i - q_i| < \frac{\delta}{\sqrt{n}}$ for $i=1,\dots,n$.

$$\Rightarrow \|\bar{x}-\bar{q}\| = \sqrt{\sum_{i=1}^{n} (x_i-q_i)^2} < \sqrt{\sum_{i=1}^{n} \frac{\delta^2}{n}} = \delta.$$

$$\Rightarrow \mathbb{Q}^n \text{ is dense } \mathbb{R}^n.$$

How is compactness related to the property of being separable?
It turns out compact $\Rightarrow$ separable for metric spaces.

Proposition 4.8.3 All compact metric spaces (X,d) are separable.
We can choose the countable dense subset A such that $\forall \delta>0$, $\exists$ finite subset A_δ of A s.t. $\forall x \in X$, $\exists a \in A_\delta$ s.t. $d(x,a)<\delta$.

Recall Proposition 3.5.12: X compact $\Rightarrow$ X is totally bounded.
$\Rightarrow \forall n \in \mathbb{N}$, $\exists$ finite # balls of radius $\frac{1}{n}$ that cover X.
Take the centers of these balls as the countable set A.
$\rightarrow$ list centers of balls with radius 1, then those with radius $\frac{1}{2}$, then those with radius $\frac{1}{3}$, etc.

Dense: Pick $n > \frac{1}{\delta}$, e.g., $n = \lceil \frac{1}{\delta} \rceil + 1$, and let

$$A_\delta = \{\text{centers of balls of radius } \tfrac{1}{n}\}.$$

$\forall x \in X$, $x \in$ some ball in A_δ, say B_x. $\rightarrow$ has radius $<\delta$

Let the center of B_x be $a \Rightarrow d(x,a) < \delta$. $\square$

We now consider how to extend these notions to $C(X,Y)$.

Recall notion of equicontinuity:

Def 4.1.3 Let (X,d_X), (Y,d_Y) be metric spaces. $\mathcal{F}$ a collection of functions $f: X \to Y$ is **equicontinuous** if $\forall \epsilon>0$, $\exists \delta>0$ s.t. $\forall f \in \mathcal{F}$, $\forall x,y \in X$ with $d_X(x,y)<\delta$ we have $d_Y(f(x),f(y)) < \epsilon$.
$\rightarrow$ Same δ for all $f \in \mathcal{F}$

Lemma 4.8.5 Let (X, d_X) is compact, and (Y, d_Y) is complete. and $\{g_k\}$ is an equicontinuous sequence in $C(X, Y)$. Let $A \subseteq X$ be a dense subset and let $\{g_k(a)\}$ converge $\forall a \in A$. Then $\{g_k\}$ converges in $C(X, Y)$.

X compact, Y complete means Theorem 4.6.2, Proposition 4.6.3 $\Rightarrow$ $C(X, Y)$ is complete, $C(X, Y) = C_b(X, Y)$.

So we check $\{g_k\}$ is Cauchy in $C(X, Y)$, i.e., $\forall \epsilon > 0$, we need to find $N \in \mathbb{N}$ s.t. $\rho(g_n, g_m) < \epsilon$ $\forall n, m \geq N$.

$\{g_k\}$ is equicontinuous $\Rightarrow \exists \delta > 0$ s.t. if $d_X(x, y) < \delta$ then $\underline{d_Y(g_k(x), g_k(y))} < \frac{\epsilon}{3}$ $\forall k \in \mathbb{N}$.

(X, d_X) is compact. So Proposition 4.8.3 $\Rightarrow$ Can choose $A_\delta \subseteq A$ s.t. $\forall x \in X$, $\exists a \in A_\delta$ s.t. $d(x, a) < \delta$.

$\{g_k(a)\}$ converges $\forall a \in A \Rightarrow \{g_k(a)\}$ is Cauchy.

$\Rightarrow \exists N \in \mathbb{N}$ s.t. $\underline{d_Y(g_n(a), g_m(a))} < \frac{\epsilon}{3}$ whenever $n, m \geq N$. $\forall a \in A_\delta$.

$\Rightarrow \forall n, m \geq N$

$$d_Y(g_n(x), g_m(x)) \leq d_Y(g_n(x), g_n(a)) + d_Y(g_n(a), g_m(a)) + d_Y(g_m(a), g_m(x))$$

triangle inequality

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

This result holds $\forall x \in X$.

$\Rightarrow \rho(g_n, g_m) < \epsilon \quad \forall n, m \geq N.$

$\Rightarrow \{g_k\}$ is Cauchy, and hence convergent in $C(X, Y)$. $\square$

We now state a couple intermediate results and the main theorem without proofs. We will solve a couple of problems using the main result — the Arzelà-Ascoli theorem (after the break!)

Corollary 4.8.7 If (X, d_X) is a compact metric space, all bounded, closed, and equicontinuous sets in $C(X, \mathbb{R}^m)$ are compact.

Lemma 4.8.8 Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\{f_n\}$ be a sequence of continuous functions $f_n : X \to Y$ that converges uniformly to $f : X \to Y$. If $\{x_n\}$ converges to $a \in X$, then $\{f_n(x_n)\}$ converges to $f(a)$.

Italian mathematicians; theorem presented in ~1890's.

Theorem 4.8.9 (Arzelà-Ascoli Theorem; AAT) Let (X, d_X) be a compact metric space. A subset $\mathcal{K}$ of $C(X, \mathbb{R}^m)$ is compact iff it is closed, bounded, and equicontinuous.