MATH 401: Lecture 28 (12/02/2025)

Today: * Arzéla-Ascoli Theorem (AAT) problems
* locally compact space

Recall: dense subset, separable space, AAT

Problem 5, LSIRA Pg 111 $f: [-1,1] \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant K if

$$|f(x)-f(y)| \leq K|x-y| \quad \forall\, x,y \in [-1,1].$$

Lipschitz continuity is defined more generally—just that this problem uses $X=[-1,1]$.

Let $\mathcal{K}$ be the set of all Lipschitz continuous functions with Lipschitz constant K such that $f(0)=0$. Show $\mathcal{K}$ is a compact subset of $C([-1,1],\mathbb{R})$.

Using AAT, we show $\mathcal{K}$ is closed, bounded, and equicontinuous.

1. Closed Let $\{f_n\}$ be a sequence in $\mathcal{K}$ that converges to f in $C([-1,1],\mathbb{R})$. Show $f \in \mathcal{K}$.

as each f_n is Lipschitz continuous

$$|f(x)-f(y)| = \lim_{n\to\infty} |f_n(x)-f_n(y)| \leq \lim_{n\to\infty} K|x-y| = K|x-y|.$$

does not depend on n

$\Rightarrow f$ is Lipschitz continuous with Lipschitz constant K.

Also, $f(0) = \lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0.$ as each $f_n \in \mathcal{K}$

$\Rightarrow f \in \mathcal{K}$.

2. <u>bounded</u> $\forall f, g \in \mathcal{K}$, we get

$$|f(x)-g(x)| \leq |f(x)-f(0)| + |\underset{=0}{f(0)} - \underset{=0}{g(0)}| + |g(0)-g(x)|$$

triangle inequality

$$\leq K|x-0| + 0 + K|0-x| = 2K|x| \leq 2K$$

as $x \in [-1,1]$.

This result holds $\forall f, g \in \mathcal{K}$, and $\forall x \in X$.

$$\Rightarrow \rho(f,g) = \sup\{|f(x)-g(x)| \mid x \in [-1,1]\} \leq 2K.$$

$\Longrightarrow \mathcal{K}$ is bounded.

3. <u>Equicontinuous</u>

$\forall \epsilon > 0$, we have

$$|f(x)-f(y)| \leq K|x-y| < K\delta = \epsilon \text{ when } \delta = \frac{\epsilon}{K}$$

and $|x-y| < \delta$.

Holds $\forall f \in \mathcal{K} \Longrightarrow \mathcal{K}$ is equicontinuous.

Since $\mathcal{K}$ is closed, bounded, and equicontinuous, $\mathcal{K}$ is compact by the AAT.

$\square$

We had seen that a subset of $\mathbb{R}^m$ is compact iff it is closed and bounded. Hence $\mathbb{R}^m$ itself is not compact. We study a less strict version of compactness for the whole metric space: local compactness.

Problem 7, LSIRA Pg 112 **Def** A metric space (X,d) is **locally compact** if $\forall a \in X$, $\exists\ \underbrace{\bar{B}(a;r)}_{\text{closed ball}}$ that is compact.

Show that $\mathbb{R}^m$ is locally compact, but $C([0,1],\mathbb{R})$ is not.

$B(\bar{a};r)$ will have $< r$

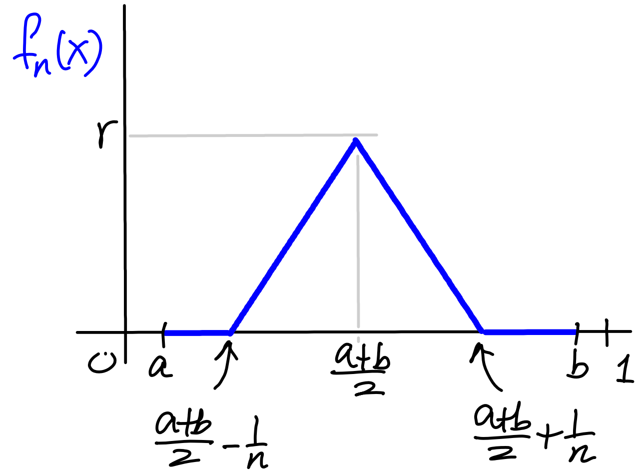
In $\mathbb{R}^m$, any $\bar{B}(\bar{a};r) = \{x \in \mathbb{R}^m \mid \|\bar{a}-\bar{x}\| \leq r\}$ is closed and bounded, and hence compact (see Corollary 3.5.5, the Bolzano-Weierstrass theorem).

$C([0,1],\mathbb{R})$: Want to show $\bar{B}(f;r)$ is not compact $\forall r$ for some continuous function $f:[0,1]\to\mathbb{R}$.

Note that we need to identify just one such function which does not satisfy the requirement, i.e., it fails for **all** balls — for all radii $r>0$.

$\bar{B}(f;r)$ is closed and bounded by definition, so by AAT need to show $\bar{B}(f;r)$ is not equicontinuous.

Consider the family of functions $f_n(x)$ defined as shown in the figure here (for $a, b \in (0,1)$). Here, $r>0$ is any radius. Note that as $n \to \infty$, $f_n(x)$ gets closer and closer to a function that is discontinuous at $x = \frac{a+b}{2}$. But each $f(x)$ is indeed continuous.



Further, we consider $f(x) = 0 \quad \forall x \in [0,1]$.

Note than $f_n(x)$ does not converge to $f(x)$ here!

We saw similar functions in LSIRA Section 4.2, e.g., see figure 4.2.1 in Page 82!

We get that $\rho(f_n, f) = \sup\{\, |f_n(x) - f(x)| \;\big|\; x \in [0,1] \,\} = r < \infty$

$\Rightarrow f_n \in \overline{B}(f, r) \quad \forall n \in \mathbb{N}$.

To have $|f_n(x) - f_n(y)| < \epsilon < r$ (for any $\epsilon > 0$) for $y = \frac{a+b}{2}$, we need to choose $\delta_n = \frac{1}{n}$ for $|x-y| < \delta$. Note that this δ_n depends on n, and hence we cannot choose the same δ that would work for all $f_n \in \overline{B}(f; r)$, as we cannot have $\delta < \frac{1}{n} \;\forall n \in \mathbb{N}$ here.

$\Rightarrow \{f_n\}$ is not equicontinuous.

$\Rightarrow \overline{B}(f; r)$ is not compact for any r, and hence $C([0,1], \mathbb{R})$ is not locally compact.